Control and synchronization of chaos in high dimensional systems: Review of some recent results

Mingzhou Ding[a)]
Center for Complex Systems and Department of Mathematical Sciences, Florida Atlantic University, Boca Raton, Florida 33431

E-Jiang Ding
Institute of Low Energy Nuclear Physics, Beijing Normal University, Beijing 100875, People's Republic of China and CCAST (World Laboratory), P.O. Box 8730, Beijing 100080, People's Republic of China

William L. Ditto
Applied Chaos Laboratory, School of Physics, Georgia Institute of Technology, Atlanta, Georgia 30332

Bruce Gluckman
Children's National Medical Center, Neurosurgery Department, 111 Michigan Avenue, N.W., Washington, D.C. 20010-2970

Visarath In
Naval Surface Warfare Center, Carderock Division, Silver Spring, Maryland 20903

Jian-Hua Peng
Department of Physics, Northeast Normal University, Changchun 130024, People's Republic of China

Mark L. Spano
Naval Surface Warfare Center, Carderock Division, Silver Spring, Maryland 20903

Weiming Yang
Center for Complex Systems and Department of Mathematical Sciences, Florida Atlantic University, Boca Raton, Florida 33431

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Controlling chaos and synchronization of chaos have evolved for a number of years as essentially two separate areas of research. Only recently it has been realized that both subjects share a common root in control theory. In addition, as limitations of low dimensional chaotic systems in modeling real world phenomena become increasingly apparent, investigations into the control and synchronization of high dimensional chaotic systems are beginning to attract more interest. We review some recent advances in control and synchronization of chaos in high dimensional systems. Efforts will be made to stress the common origins of the two subjects. © *1997 American Institute of Physics.* [S1054-1500(97)00304-2]

Chaotic phenomena arise ubiquitously in natural systems and in man-made devices. Past work has focused mainly on the discovery and characterization of chaotic behavior occurring in situations where there is no goal-oriented intervention, i.e., control. Recently, ideas and techniques have been proposed to utilize the rich properties of chaos to achieve certain objectives. One idea is to convert chaotic orbits to desired periodic ones by using temporally programmed small controls. It is suggested that by doing so one improves the system's performance against some general classes of criteria. Another idea explores the synchronization of chaotic systems. The intended application here is secure communication. In this work we consider these ideas in a unified framework.

I. INTRODUCTION

The year 1990 saw the publication of two seminal papers, *Controlling Chaos* by Ott, Grebogi and Yorke (OGY),[1] and *Synchronization in Chaotic Systems* by Pecora and Carroll.[2] Promising wide applications outside the traditional scope of chaos and nonlinear dynamics research, these two papers immediately received a great deal of attention, and have led to the establishment of two active areas of research.[3]

Although it was known from the start that the techniques of chaos control have their origins in control theory, synchronization of chaos has evolved somewhat in its own right. Recent progress[4] casts the problem of chaos synchronization in the framework of nonlinear control theory. This unifies the study of chaos control and chaos synchronization under the same rubric as the title of this special focus issue suggests.

Since 1990 many hundreds of papers have been written on the control and synchronization of chaos. It is fair to say, however, that the majority of these publications deal with low dimensional chaotic systems, representative examples of which include the Rössler system or the Lorenz system. Specifically, in the case of chaos control, the stabilized periodic orbit is often characterized by one unstable and one stable direction, while for chaos synchronization, the synchronized

[a)]Electronic mail: ding@walt.ccs.fau.edu

attractor admits only one positive Lyapunov exponent. This situation is not very satisfying since it is known that low dimensional systems have rather limited usefulness in modeling real world applications. Thus we must be able to carry out chaos control and achieve synchronization of chaotic systems in high dimensions. The goal of this paper is to review some recent results in this endeavor.

The remainder of this paper is divided into two parts, Part I on high dimensional chaos control and Part II on high dimensional chaos synchronization. We will make attempt to present both topics in a unified framework.

II. PART I. CONTROLLING CHAOS IN HIGH DIMENSIONS

A. Introduction to Part I

The OGY control method[1] is designed for stabilizing a periodic orbit that has effectively one stable and one unstable direction. It requires no knowledge of the underlying equations of motion and assumes only the availability of a measured time series from the system and a control parameter that can be manipulated. The phase space needed for control is reconstructed from data by the use of delay embedding techniques. A salient feature of the OGY type of control approach is that, since the periodic orbit to be stabilized is part of the system's natural dynamics, control is often achieved by applying only small temporally programmed perturbations to the accessible control parameter.

Soon after the OGY technique appeared in the literature, Ditto, Rauseo, and Spano[5] managed to control the chaotic dynamics experimentally in a periodically driven magnetoelastic ribbon system. This work firmly established the viability of chaos control in practical applications and, together with the OGY paper, inspired a large body of subsequent theoretical and experimental studies. For a sample see Refs. 6–12.

From the theoretical side Dressler and Nitsche made an important observation in 1992.[8] Let us assume that for the given system the dynamics is described by a map (e.g., on the Poincaré surface of section). To achieve and effectively maintain control one perturbs a system parameter at every iteration of the map. This implies that the map actually changes from one iteration to the next. Dressler and Nitsche proposed to address this problem by modifying the OGY control to include the dynamics of the control parameter. With the new method they are able to achieve control in systems where the ordinary OGY fails.

The modified OGY method by Dressler and Nitsche is still two dimensional. High dimensional methods incorporating the dynamics of the control parameter appeared in Refs. 9–11. Specifically, the paper by So and Ott,[9] in which they introduce the notion of a system-plus-parameter dynamical system, is a generalization of Romeiras *et al.*[12] high dimensional control technique. Petrov *et al.*[10] base their method on the control engineering idea of a single-input-single-output design. Ding *et al.* technique[11] is based on that of So and Ott. Assuming Poincaré map generated time series, they are able to derive explicit control laws that are more amenable to experimental implementation, as indeed demonstrated by their successful control of a high dimensional experimental system. To illustrate the main ideas involved in this type of control approach we present below an abbreviated derivation of the control technique of Ding *et al.* and some accompanying numerical results.

B. The control method

Assume that the dynamical system of interest is described by the following k-dimensional map on some Poincaré surface of section,

$$\mathbf{X}_{n+1}=\mathbf{F}(\mathbf{X}_n,p), \tag{1}$$

where $\mathbf{X}\in\mathbf{R}^k$ and p is the control parameter to be perturbed. Suppose that for $p=p^*$ Eq. (1) has a chaotic attractor. Without explicitly knowing $\mathbf{F}$ we base our analysis and control on a discretely measured time series $\{x_n\}$ of some scalar observable $x_n=h(\mathbf{X}_n)$. Using delay coordinates[13] one reconstructs the high dimensional dynamics from $\{x_n\}$ via, $\mathbf{z}_n=(z_n^{(1)},z_n^{(2)},\cdots,z_n^{(m)})^T=(x_{n-m+1},x_{n-m+2},\cdots,x_n)^T$, where m is the dimension of the reconstructed phase space and T denotes matrix transpose. For large enough m, $\mathbf{z}_n$ is a global one-to-one representation of the variable $\mathbf{X}_n$ on the original attractor. Since the control is done by changing the value of p according to a control law for every iteration of the map, the reconstructed discrete map for $\mathbf{z}_n$ has the form,

$$\mathbf{z}_{n+1}=\mathbf{G}(\mathbf{z}_n,p_{n-m+1},p_{n-m+2},\ldots,p_n). \tag{2}$$

Here $\mathbf{G}$ generally depends on all the parameter variations effective during the time interval $n-m+1\leqslant t\leqslant n$ spanned by the delay vector $\mathbf{z}_n$.[8–10]

In the reconstructed phase space a fixed point for the nominal system (i.e., when $p=p^*$) is denoted by $\mathbf{z}^*=\mathbf{G}(\mathbf{z}^*(p^*),p^*,p^*,\cdots,p^*)$. The Jacobian matrix for Eq. (2) is the following $m\times m$ matrix[9]

$$\mathbf{A}_n=\mathbf{D}_{\mathbf{z}_n}\mathbf{G}(\mathbf{z}_n,p_{n-m+1},p_{n-m+2},\cdots,p_n).$$

The set of m-dimensional column vectors,

$$\mathbf{B}_n^{(i)}=D_{p_{n-i+1}}\mathbf{G}(\mathbf{z}_n,p_{n-m+1},p_{n-m+2},\cdots,p_n),$$

for $i=1,2,\ldots,m$, characterize the effect of the control parameter variations on the dynamics. Evaluating all the partial derivatives at $\mathbf{z}_n=\mathbf{z}^*(p^*)$ and $p_{n-m+1}=p_{n-m+2}=\cdots=p_n=p^*$, one obtains the linearized dynamics around the fixed point,[9]

$$\begin{aligned}\delta\mathbf{z}_{n+1}=\mathbf{A}\delta\mathbf{z}_n+\mathbf{B}^{(m)}\delta p_{n-m+1}+\mathbf{B}^{(m-1)}\delta p_{n-m+2}\\ +\cdots+\mathbf{B}^{(1)}\delta p_n,\end{aligned} \tag{3}$$

where $\delta\mathbf{z}_i=\mathbf{z}_i-\mathbf{z}^*$, $\delta p_i=p_i-p^*$ and the reference to n for $\mathbf{A}$ and $\mathbf{B}$ has been dropped since they are constant at the fixed point. Note that, due to the nature of the time series and delay coordinates used here, most of the entries in the above matrix and vectors are zero. Specifically, one has[11]

$$\mathbf{A}=\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ a_m & a_{m-1} & a_{m-2} & \cdots & a_1 \end{pmatrix}_{m\times m}, \quad \mathbf{B}^{(i)}=\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b_i \end{pmatrix}_{m\times 1}, \tag{4}$$

where $i=1,2,\cdots,m$.

It has been pointed out in the past[8] that it is undesirable to derive control laws based directly on Eq. (3). Following So and Ott,[9] Ding *et al.*[11] introduce a $2m-1$ dimensional expanded phase space, $\mathbf{Y}_n=(x_{n-m+1},x_{n-m+2},..,x_n, p_{n-m+1},p_{n-m+2},..,p_{n-1})^T$, to accommodate both dynamical measurements x_i and parameter changes p_i. In this expanded phase space, near the unstable fixed point $\mathbf{Y}^*=(x^*,\cdots,x^*,p^*,\cdots,p^*)^T$, the linearized dynamics becomes,

$$\mathbf{Y}_{n+1}-\mathbf{Y}^*=\tilde{\mathbf{A}}(\mathbf{Y}_n-\mathbf{Y}^*)+\tilde{\mathbf{B}}(p_n-p^*) \tag{5}$$

with

$$\tilde{\mathbf{A}}=\begin{pmatrix} \mathbf{A} & \mathbf{B}^{(m)} & \mathbf{B}^{(m-1)} & \mathbf{B}^{(m-2)} & \cdots & \mathbf{B}^{(2)} \\ \mathbf{0} & 0 & 1 & 0 & \cdots & 0 \\ \mathbf{0} & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & 0 & 0 & 0 & \cdots & 1 \\ \mathbf{0} & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}_{(2m-1)\times(2m-1)}, \quad \tilde{\mathbf{B}}=\begin{pmatrix} \mathbf{B}^{(1)} \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}_{(2m-1)\times 1}, \tag{6}$$

where $\mathbf{0}$ indicates an m-dimensional row vector of 0's.

To control one applies a suitable perturbation $\delta p_n=p_n-p^*$, following each measurement x_n, to keep the dynamics within the stable subspace of $\tilde{\mathbf{A}}$. For a fixed point with u unstable directions, the control law governing the choice of δp_n is derived to be[11]

$$\delta p_n=-\left(\sum_{k=1}^{u}\frac{(\lambda_k)^u}{(\mathbf{v}_k^T\tilde{\mathbf{B}})\prod_{i=1,i\neq k}^{u}(\lambda_k-\lambda_i)}\mathbf{v}_k^T\right)\delta\mathbf{Y}_n, \tag{7}$$

where λ_k are the unstable eigenvalues of $\tilde{\mathbf{A}}$, ordered in descending absolute value, and the contravariant unstable eigenvectors $\mathbf{v}_k$ are defined by $\tilde{\mathbf{A}}^T\mathbf{v}_k=\lambda_k\mathbf{v}_k$. It can be shown[11] that the elements of $\mathbf{v}_k=(v_k^{(1)},v_k^{(2)},\cdots,v_k^{(2m-1)})$ are $v_k^{(i)}=\Sigma_{j=1}^{i}a_{m-j+1}(\lambda_k)^{j-i-1}$ for $i<m$, $v_k^m=1$, and $v_k^{(i)}=\Sigma_{j=1}^{i-m}b_{m-j+1}(\lambda_k)^{j+m-i-1}$ for $i>m$.

A more general method of stabilizing a period-N orbit along the same line is given in Ref. 11. In the case where the target fixed point has large eigenvalues one needs to make multiple (e.g., N) measurements and controls in between two proper Poincaré surface of sections to minimize the impact of the large eigenvalues (see the paper by Christini *et al.* in Ref. 12). A fixed point would appear to be a period-N orbit in this case. The period-N control method can be applied to stabilize a fixed point by perturbing the control parameter every time one makes an observation of the system state.

C. Time series formation and numerical results

Time Series Formation. In experimental problems, one expects to encounter two classes of continuous-time systems, autonomous and periodically driven. The question of how to form discrete time series in both cases so that the control laws developed in Section II B for discrete maps are directly applicable is addressed below.

First, consider an autonomous system defined as

$$d\mathbf{Z}/dt=\mathbf{G}(\mathbf{Z},p),$$

where $\mathbf{Z}\in\mathbf{R}^{k+1}$. Let $x=h(\mathbf{Z})$ denote a scalar observable function. Consider the plot of x versus t. Ding *et al.*[11] introduce the method to form the discrete time series by measuring the intervals I_n between the $(n-1)$th and nth upward (or downward) crossings of some predetermined threshold $x=x_c$ (see Fig. 1). It can be argued that these variables I_n, which are called interspike intervals, sample the dynamics of some Poincaré map in the original $\mathbf{Z}$ phase space.[14] Specifically, at each crossing in the x versus t plot, the condition $x=h(\mathbf{Z})=x_c$ is met. This condition defines a k-dimensional Poincaré surface of section in the original $\mathbf{Z}$ space. Thus, I_n is also the time between the $(n-1)$th and the nth crossings of the section. Suppose one parametrizes this section by a k-dimensional vector $\mathbf{Q}$. Then, the successive crossings of the plane from a given direction by a chaotic trajectory give rise to a Poincaré map,

$$\mathbf{Q}_{n+1}=\mathbf{P}(\mathbf{Q}_n,p). \tag{8}$$

The interval I_n is uniquely determined by $\mathbf{Q}_{n-1}$, namely,

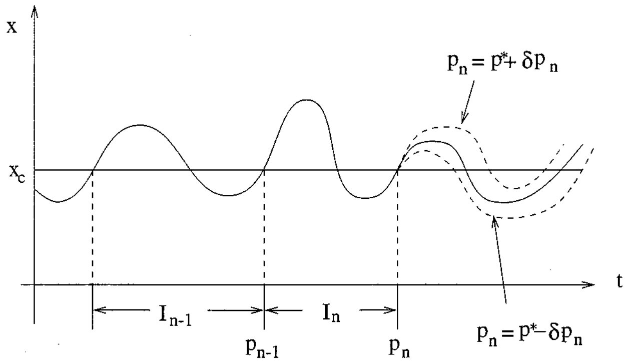

FIG. 1. Schematic illustrating the formation of interspike intervals from a continuous time series and the effect of control on the intervals.

$$I_n = \Phi(\mathbf{Q}_{n-1}). \tag{9}$$

One can view I_n as a scalar measurement variable of the high dimensional dynamics of $\mathbf{Q}_{n-1}$.

Traditionally, one measures the continuous time series x versus t using equally spaced sampling intervals. In the reconstructed phase space one obtains the Poincaré map by examining the crossings of some plane by the reconstructed trajectory. Due to the discreteness of the trajectory one encounters inevitable errors in the resulting Poincaré map through interpolation.[14] In contrast, Ding *et al.*[11] way of forming the discrete time series using interspike intervals avoids this problem by monitoring the analog signal and thus detects the threshold crossing precisely. Furthermore, the reconstructed interspike intervals already obey a Poincaré map. The effect of parametric perturbations on the interspike intervals is illustrated in Fig. 1.

Next, consider a periodically forced system

$$d\mathbf{Z}/dt = \mathbf{G}(\mathbf{Z},t,p),$$

where $\mathbf{Z} \in \mathbf{R}^k$ and $\mathbf{G}(\mathbf{Z},t+T,p) = \mathbf{G}(\mathbf{Z},t,p)$. Let $x = h(\mathbf{Z})$ be the scalar observable function. Since, by introducing t as an additional variable one can convert a nonautonomous system to an autonomous one, the interspike intervals formation method also applies here. Another more traditional method of forming a discrete time series $\{x_n\}$ is by measuring x at times $t_n = nT + T_0$ (stroboscopic sampling). From the theorems of Ref. 13, the dynamics reconstructed from $\{x_n\}$ in a suitable delay coordinates space represents the dynamics of the Poincaré map, $\mathbf{Z}_{n+1} = \mathbf{P}(\mathbf{Z}_n)$, in the original phase space which, in turn, is equivalent to the continuous-time dynamics described by the differential equations.

Example of Control. Consider the following four dimensional autonomous chemical reaction model[15]

$$\begin{aligned}
\dot{x} &= pw/(1+w^{10}) - 0.1x,\\
\dot{y} &= 0.1x - 0.2yz,\\
\dot{z} &= 0.2z(y-w),\\
\dot{w} &= 0.2zw - 0.1w.
\end{aligned} \tag{10}$$

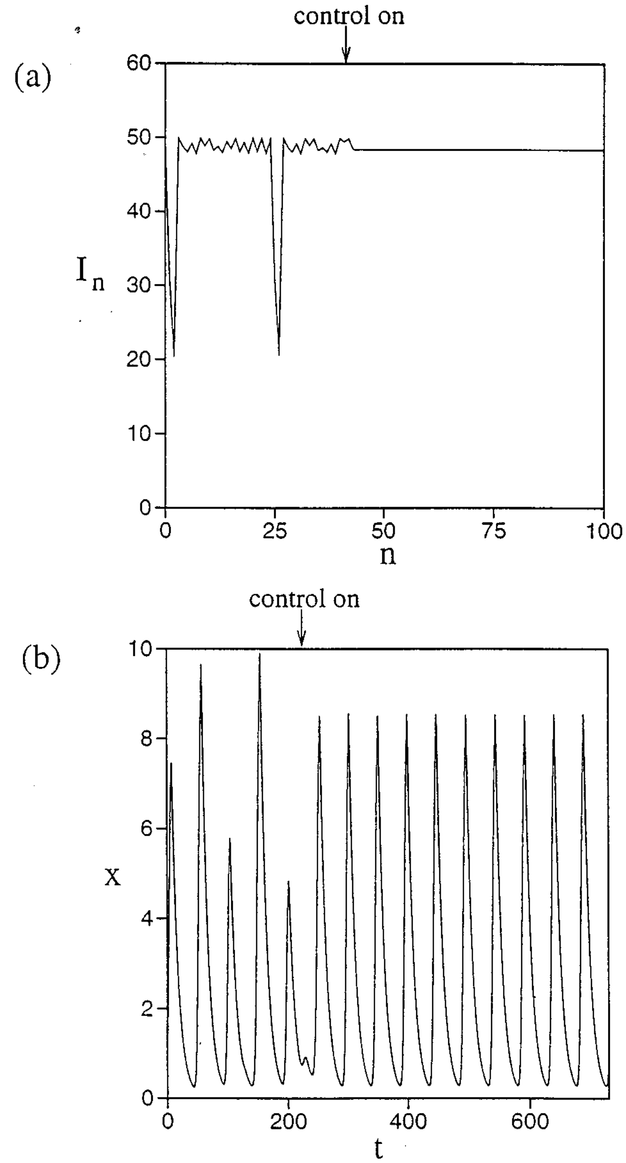

FIG. 2. Results of controlling the unstable period-1 orbit in the chemical reaction model Eq. (10) using an $m=3$ implementation of the control. (a) The interspike intervals and (b) the corresponding continuous time series.

For $p = p^* = 2.5$ this system exhibits a chaotic attractor of dimension $D = 2.2$. Assume that the observed scalar variable is x itself. By measuring the times between successive upward crossings of some threshold x_c by the x versus t function one forms the interspike interval time series $\{I_n\}$. Reconstructing this discrete time series in an $m=$ three dimensional delay coordinate space one obtains an attractor of dimension 1.2. In this attractor there is a fixed point $(N=1)$ at $\mathbf{z}_n = (48.40,48.40,48.40)^T$.[11]

Consider the control of the fixed point. Set $x_c = 1$. From the numerically generated time series one obtains $a^{(1)} = -1.33$, $a^{(2)} = 0.41$, $a^{(3)} = -0.008$ and $b^{(1)} = -1.25$, $b^{(2)} = -3.20$, $b^{(3)} = 1.95$. The unstable eigenvalue is calculated to be $\lambda_1 = -1.59$. The result of applying Eq. (7) is shown in Fig. 2[11] where the system behavior before and after the control is turned on is displayed. Figure 2(a) is the interspike intervals and Fig. 2(b) gives the corresponding continuous time series.

D. Adaptive control of chaos in high dimensions

The above control approach makes the tacit assumption that the environment of the system remains static. This is seldomly true in practice. It is often the case that changing ambient physical conditions, entering the system as parameters, cause change in the dynamics in an irregular and unpredictable fashion. If small perturbation control is to be maintained one must be able to continually update both the periodic orbit position and the other control parameters. This adaptive control or tracking approach is pioneered by Schwartz and Triandaf[16] in the context of controlling chaos. Other works can be found in Ref. 17. Below we review recent results reported in Ref. 18 where the control and tracking of period orbits in high dimensions is the main focus. We remark that various scenarios have been considered in which a momentary loss of control due to environmental variations and the lack of adaptive control leads to catastrophic system failures (see Ding *et al.* paper in Ref. 7). In addition, by applying adaptive control one can extend the stable operation range of a given system.

Adaptive Control Strategy. In terms of the measured variable x and the parameter p, Eq. (3) can be written as

$$x_{n+1}-x^*=\sum_{\alpha=1}^{m} a_\alpha(x_{n-\alpha+1}-x^*)+\sum_{\beta=0}^{m} b_\beta \delta p_{n-\beta+1}. \tag{11}$$

The quantities a_α, b_β, and x^* in the above difference equation fully determine the control law in Eq. (7). In fact, a major step to achieve initial control is to find ways to fit the unperturbed and perturbed dynamics near the fixed point to obtain the values of these parameters. This is usually done with the least squares technique. See Ref. 11. Here we suppose that this step has been completed. Assume now that the system parameters have changed slightly such that the fixed point position and the shape of the linear region are modified, but not so much that the control is lost, nor that the dynamics during control are outside the linear region of the new fixed point. In this case, Eq. (11) still applies, and we can use a short history of N points incorporating the most recent measurements of the system state and perturbations to refit the control parameters. This refitting can be repeated as often as once per measurement cycle and requires no *a priori* knowledge or assumptions about how the environment has changed.

Below we explain Gluckman *et al.*[18] strategy on how to update the fix point position x^* and the parameters a's and b's in two separate steps, each fitting step incorporating a different idea. In practice these steps can be combined into one step.

(1) Assume that a's and b's remain constant for the duration of N iterates. From the expression in Eq. (11), we obtain

$$\langle x_{n+1}-x^*\rangle_N=\sum_{\alpha=1}^{m} a_\alpha\langle x_{n-\alpha+1}-x^*\rangle_N +\sum_{\beta=1}^{m} b_\beta\langle \delta p_{n-\beta+1}\rangle_N, \tag{12}$$

where $\langle\rangle_N$ denotes average over the history of N measurements. Upon rearranging terms one has

$$\langle x^*\rangle_N=\langle x_n\rangle_N+\frac{\Sigma_\beta b_\beta}{1-\Sigma_\alpha a_\alpha}\langle \delta p_n\rangle_N=\langle x_n\rangle_N+g\langle \delta p_n\rangle_N, \tag{13}$$

where

$$g=\frac{\Sigma_\beta b_\beta}{1-\Sigma_\alpha a_\alpha}. \tag{14}$$

At the end of this fitting cycle, $\langle x^*\rangle_N$ is taken as the new fixed point position. Although this method of calculating x^* may seem simplistic, it by construction guards against a nonzero average in δp. Therefore its use maintains the small-perturbation quality of the control.

(2) Assuming now that x^* is a constant, for N iterates, we apply a least squares fit to Eq. (11) to get the values of a's and b's. To do so accurately, one must enforce a small amount of motion about the fixed point, because incessant applications of control suppress the motion around the fixed point, making the estimation of the a's and b's difficult. (This technique is called interrogation by Petrov *et al.*[19]) It is worthwhile to note that control can often be maintained for a while during the environment change without having to update the a's and b's. However, when the control fails as a result of the actual fixed point being too far removed from its original known location, it is often too late to update the a's and b's. By enforcing a small amount of natural local dynamics we are able to update the a's and b's more precisely and more frequently. Specifically, this is done as follows. We monitor the absolute value of the perturbation δp_n. If it is smaller than some predetermined threshold $\delta p_{\min}$ we do not apply the perturbation (i.e., setting $\delta p_n=0$) and let the system evolve freely for that iterate. This proves effective in Gluckman *et al.* numerical and experimental work.[18]

The estimated values of x^* and a's and b's are subject to statistical fluctuations that can be very significant at times. As a precaution Gluckman *et al.*[18] test the veracity of the newly derived parameters by checking whether the new values are close to the previous values. In fitting x^*, one defines a maximal distance between the newly estimated value and the previously used value. If the measured distance exceeds this maximal distance one does not update the fixed point position. In testing both the a's and the b's one relies on quantities derived from a's and b's. Since the a's describe the unperturbed dynamics near the unstable fixed point, the critical information they contain for control purposes are the directions of the stable and unstable manifolds of the fixed point. As a test of the fit of the a's Gluckman *et al.*[18] place a maximal angular deviation of, say, 15° between the new most unstable direction and the previous one. In order to test the new values of the b's one limits the fractional difference in the quantity g defined in Eq. (14). Specifically, in the experimental work reported in Ref. 18, they require

$$\frac{(g_{\text{calc}}-g_{\text{old}})^2}{|g_{\text{calc}}*g_{\text{old}}|}<2.25. \tag{15}$$

We comment however that for a given system one should tailor the criteria to achieve the best result.

Numerical Example. Gluckman *et al.*[18] applied the tracking technique above to a coupled driven Duffing oscillator expressed as

$$\begin{aligned}\ddot{x}_1+\gamma\dot{x}_1+\alpha(x_1^3-x_1)+\beta_1(x_1-x_2)&=p_1\sin(\omega t),\\ \ddot{x}_2+\gamma\dot{x}_2+\alpha(x_2^3-x_2)+\beta_2(x_2-x_1)&=p\sin(\omega t).\end{aligned} \tag{16}$$

This oscillator, probed at the integer multiples of the external driving period, is described by a four dimensional discrete map. The key point here is that the periodic orbit to be stabilized and followed has two unstable directions.

For $\gamma=0.632$, $\alpha=4.0$, $\beta_1=0.1$, $\beta_2=0.05$, $\omega=2.1235$, $p_1=1.011$, and $p=p^*=p_1$, Eq. (16) exhibits a chaotic attractor of dimension $D=3.3$. The scalar observable here is $x=x_1+x_2$ and the attractor is sampled every cycle of the external forcing. The attractor reconstructed using the time series $\{x_n\}$ in an m- four dimensional delay coordinates space has a dimension $D=2.3$.

The reconstructed attractor contains a fixed point at $(0.54,0.54,0.54,0.54)^T$. This fixed point corresponds to the synchronized period one motion of the coupled oscillators and has two unstable directions ($u=2$). This orbit was originally stabilized in Ref. 11 using Eq. (7) with $m=4$. Let the environment, represented by the value of α, drift slowly over time as shown in Fig. 3(a). With p as the control parameter Gluckman *et al.*[18] were able to to maintain control by applying the adaptive control strategy developed earlier. The position of the tracked fixed point is shown in Fig. 3(b). $N=20$ was in Eq. (13) for updating both the fixed point position and the a's and b's. Again, although the theoretical value of the fixed point position is not known, it is clear from the figure that its movement is compatible with the movement of α. Figure 3(c) show the values of the first two eigenvalues of the tracked fixed point. For the most part both eigenvalues have magnitude greater than one.

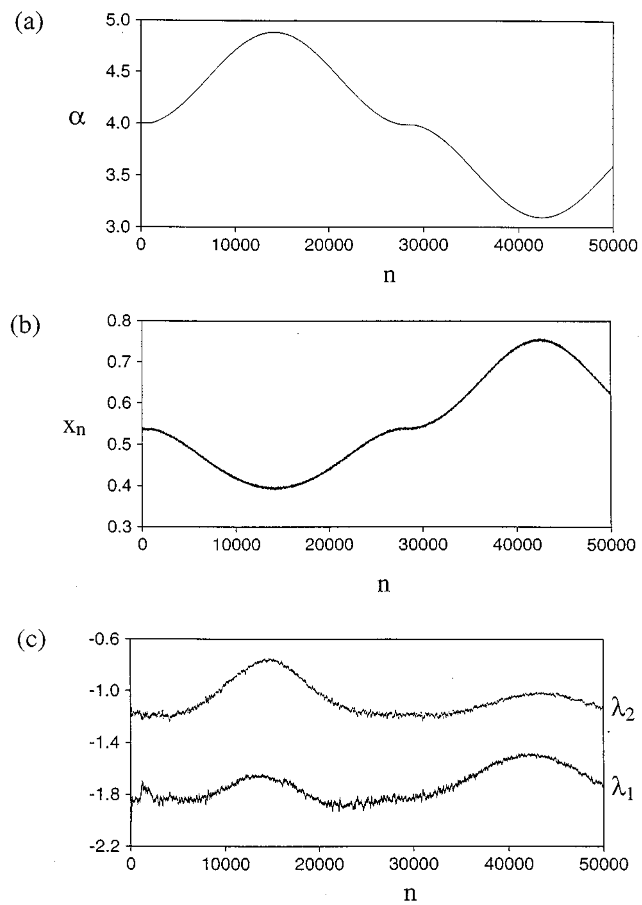

FIG. 3. (a) Slow drift of the parameter α in the coupled Duffing oscillator Eq. (16). (b) Measured variable $x=x_1+x_2$ during control and tracking. (c) The first two eigenvalues of the tracked fixed point.

III. PART II. SYNCHRONIZING CHAOS IN HIGH DIMENSIONS

A. Introduction to Part II

Chaotic trajectories are locally unstable. This instability, known as the sensitive dependence on initial conditions, is quantified by the system's Lyapunov exponents. A system having one positive Lyapunov exponent is said to be chaotic, and a system having more than one positive Lyapunov exponents is said to be hyperchaotic. Clearly, two identical chaotic systems cannot synchronize in isolation since the slightest difference in their respective initial conditions will lead to two totally different trajectories.

In 1990 Pecora and Carroll[2] made an important observation. They found that, when you make a replica of part of a chaotic system and send a system variable from the original system (sender) to drive this replica (receiver), sometimes the replica subsystem and the original chaotic one can lock in their steps and evolve together chaotically in synchrony. They further propose that this chaos synchronization phenomenon may serve as the basis for new ways to achieve secure communications. This paper has inspired a great deal of subsequent work[4,20–28] comprising the current field of chaos synchronization.

A breakthrough came in 1993 when Cuomo and Oppenheim[21] proposed a chaos masking scheme for transmitting secret messages. Their basic idea is to add much louder chaotic noise to a message bearing signal. An eavesdropper only hears the chaos which sounds like noise. The intended recipient of the message, equipped with a prefabricated receiver based on the principle of chaos synchronization, can reproduce the chaos, subtract it out from the composite signal and recover the message. This elegant approach has one drawback. That is, even in ideal cases, the synchronization is not exact. Thus the recovered message only approximates the original message. This is overcome by Wu and Chua[22] who make essential modifications to the original Cuomo and Oppenheim scheme.

Subsequent research reveals another problem with the idea of achieving secure communication via chaos synchronization. This has to do with the fact that, the models people have often investigated in this area are primarily low dimensional systems with one positive Lyapunov exponent. Short[23] and Perez and Cerdeira[24] showed that, when such simple chaotic processes are used to mask a message, the hidden message can often be retrieved easily by an eavesdropper without having to have the receiver. Their unmasking procedure involves intercepting the composite signal as a time series and reconstructing it in a two dimensional space using delay coordinates. Since the masking chaotic system is simple one can learn a great deal about its properties by just studying this two dimensional delay coordinate plot. Once enough is learned about the chaotic process one subtracts it out and recovers the message without any explicit knowledge about the original masking system.

One way to deal with this security problem is to use high dimensional systems with multiple positive Lyapunov exponents (hyperchaos) as masking processes.[25] This type of process increases security by giving rise to much more complex time series which are not vulnerable to the unmasking procedure employed by Short[23] and by Perez and Cerdeira[24] and other code breaking methods. But questions arise as to whether by transmitting just a single variable, as would be desired in a communication situation, one could achieve synchronization between hyperchaotic systems. There is in fact a generally held belief that one needs to simultaneously transmit as many variables as the number of positive Lyapunov exponents to account for the equal number of unstable eigendirections. If this was the case, then the very purpose of using chaos synchronization as a tool for secure communication would have been defeated, since most communication schemes are best done by broadcasting just one signal.

In this part of the paper we begin by reviewing a result of Peng *et al.*[26] which proves the above belief to be incorrect. They show that, for hyperchaotic systems, chaos synchronism is attained over a broad range of parameters by using a transmitted signal that is expressed as a linear combination of the original phase space variables. Peng *et al.*[26] approach is further improved by the work of Grassi and Mascolo[28] who consider a special class of dynamical systems from the viewpoint of nonlinear observer theory. These results enable the utilization of hyperchaotic systems and at the same time preserve the simplicity of the original Pecora-Carroll approach.

B. Synchronizing hyperchaos by a scalar signal

Problem Formulation. Let us assume the sender to be a chaotic system in the form

$$d\mathbf{x}(t)/dt = \mathbf{F}(\mathbf{x}(t)), \tag{17}$$

where $\mathbf{x} \in \mathbf{R}^m$ is an m-dimensional vector. In Ref. 26 Peng *et al.* take the signal to be transmitted as a scalar variable in the form $u(t) = \mathbf{K}^T\mathbf{x}(t) = K_1x_1(t) + K_2x_2(t) + \cdots + K_mx_m(t)$, with $\mathbf{K}$ a constant column vector. Here T denotes matrix transpose. The receiver subsystem is then written as

$$d\mathbf{y}(t)/dt = \mathbf{F}(\mathbf{y}(t)) - \mathbf{B}(v(t) - u(t)), \tag{18}$$

where $v(t) = \mathbf{K}^T\mathbf{y}(t)$ and $\mathbf{B} = (B_1, B_2, \cdots, B_m)^T$ is an m-dimensional constant vector.

Observe that if $\mathbf{y}(t) = \mathbf{x}(t)$ is plugged into Eq. (18), the equation is satisfied, meaning that synchronization of chaos is possible for the combined system, Eqs. (17) and (18). The question is whether this solution is attracting so that it can be realized in practice, or in other words, whether Eq. (18) is an observer of Eq. (17). To answer this question one evaluates the largest Lyapunov exponent[27] for the system Eq. (18) with respect to the trajectory $\mathbf{y}(t) = \mathbf{x}(t)$. Consider infinitesimal deviations of $\mathbf{y}(t)$ from $\mathbf{x}(t)$, i.e.,

$$\mathbf{y}(t) = \mathbf{x}(t) + \delta\mathbf{y}(t).$$

From Eq. (18)

$$d\,\delta\mathbf{y}/dt = (\partial\mathbf{F}(\mathbf{y})/\partial\mathbf{y}|_{\mathbf{y}=\mathbf{x}} - \mathbf{B}\mathbf{K}^T)\,\delta\mathbf{y}. \tag{19}$$

The largest subsystem Lyapunov exponents, denoted Λ, is then given by solving Eq. (19) using a typical orbit $\mathbf{x}(t)$ for the original system (17) and a typical orientation for $\delta\mathbf{y}(0)$,

$$\Lambda = \lim_{t\to\infty}\frac{1}{t}\ln\frac{|\delta\mathbf{y}(t)|}{|\delta\mathbf{y}(0)|}. \tag{20}$$

When $\Lambda < 0$, for typical $\mathbf{y}(0) \neq \mathbf{x}(0)$ in the basin of attraction of the synchronization attractor,

$$\lim_{t\to\infty}|\mathbf{y}(t) - \mathbf{x}(t)| = 0. \tag{21}$$

Namely, synchronism takes place.

Note that it is a common practice to transmit a signal that is a phase space variable of the original system. It is found that synchronism is often not attained when this is the case. The problem becomes more acute when the original chaotic system has two or more positive Lyapunov exponents. For such systems, even the use of a nonreplica receiver subsystem as proposed in Ding and Ott[20] which has tunable parameters in the $\mathbf{B}$ vector, can become inadequate. In this regard, Peng *et al.*[26] approach outlined above represents an important step in methodology toward remedying the situation. In particular, Λ in Eq. (20) is now a function of the $2m$ parameters contained in both $\mathbf{K}$ and $\mathbf{B}$ vectors. This enlarged **K**-**B** space renders greater flexibility not only in designing the characteristics of the transmission but also in choosing the correct parameter combinations to meet the condition for and enhance the performance of synchronism. Specifically, one's task is now reduced to that of locating a suitable region in the **K**-**B** space for which Λ is negative. This is illustrated for the hyperchaotic Rössler equation.

Numerical Example. The hyperchaotic Rössler[29] system treated by Peng *et al.*[26] is written as

$$dx_1/dt = -x_2 - x_3, \tag{22}$$

$$dx_2/dt = x_1 + 0.25x_2 + x_4, \tag{23}$$

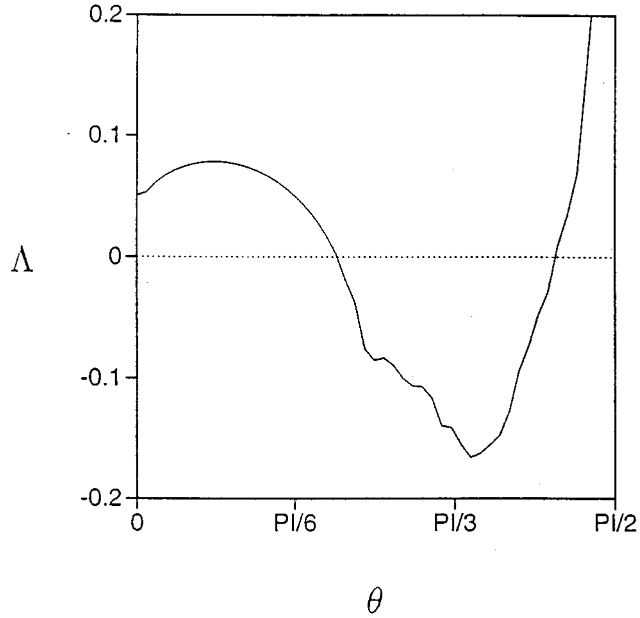

FIG. 4. The largest subsystem Lyapunov exponent Λ for the Rössler system Eqs. (22)–(25) plotted as a function of the parameter θ. It is seen that over a substantial range of the θ value Λ is negative indicating synchronism between the sender and the receiver systems.

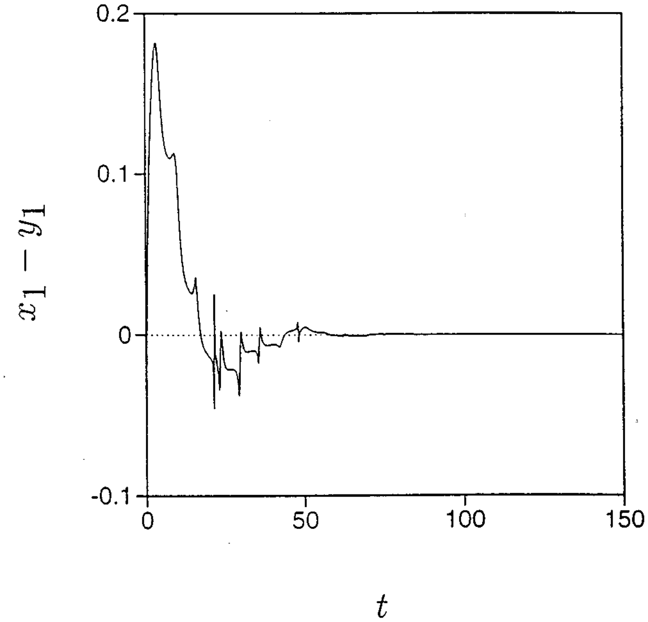

FIG. 5. Result from our numerical synchronization experiment using $\theta = \pi/3$ (see Fig. 4). The difference between the variable x_1 of the sender and the variable y_1 of the receiver asymptotes zero as time progresses.

$$dx_3/dt = 3.0 + x_1 x_3, \tag{24}$$

$$dx_4/dt = -0.5x_3 + 0.05x_4. \tag{25}$$

Numerical evidence indicates that the attractor has two positive Lyapunov exponents $\lambda_1 = 0.11$ and $\lambda_2 = 0.02$. Choosing $\mathbf{K} = (\sin\theta, 0, \cos\theta, 0)^T$ and $\mathbf{B} = 5.0(\cos\theta, 0, \sin\theta, 0)^T$ somewhat arbitrarily, Peng *et al.*[26] plot the largest receiver subsystem Lyapunov exponent Λ as a function of θ in Fig. 4. As can be seen, there is a range of θ values over which Λ is negative. In Fig. 5 the result of the synchronization experiment performed on the Rössler system is shown by plotting the difference between x_1 and y_1 for $\theta = \pi/3$. Within the resolution of the figure one obtains chaos synchronism in about 60 time units. Note that in this example, when a plain phase space variable (i.e., x_i, $i = 1,2,3,4$) is sent as the input to the receiver subsystem, no synchronization is observed.

Clearly, for this approach to be successful one needs to resolve the question concerning how to find regions in the **K**-**B** parameter space yielding negative Λ. Peng *et al.*[26] propose the following general method as a possible solution. First, for a given system, replace in Eq. (19) the vector **x** and its functions by their average values calculated on the original chaotic attractor. Now Eq. (19) becomes a linear equation of constant coefficients. Second, solve this linear equation and study how the eigenvalues change as the parameters B_i and K_i vary. This will give a rough indication as to how to locate a favorable region to start our search. Third, beginning in a favorable region obtained above, integrate the full Eqs. (17) and (19), and gradually expand the parameter space exploration until finding a region where the value of Λ is negative.

Nonlinear Observer Design and Hyperchaos Synchronization. For a special class of dynamical systems Grassi and Mascolo[28] made the following important observation. Assume that **F** in Eq. (17) has the following form

$$\mathbf{F}(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{C}f(\mathbf{x}) + \mathbf{D}, \tag{26}$$

where **A** is an $m \times m$ matrix, $f(\mathbf{x})$ is a scalar function, and **C** and **D** are m-dimensional column vectors. Equation (17) now can be written as

$$d\mathbf{x}(t)/dt = \mathbf{A}\mathbf{x} + \mathbf{C}f(\mathbf{x}) + \mathbf{D}. \tag{27}$$

Let the transmitted scalar signal be

$$u(t) = f(\mathbf{x}) + \mathbf{K}^T\mathbf{x},$$

and express the receiver system as

$$d\mathbf{y}(t)/dt = \mathbf{A}\mathbf{y} + \mathbf{C}f(\mathbf{y}) + \mathbf{D} - \mathbf{C}(v(t) - u(t)), \tag{28}$$

where $v(t) = f(\mathbf{y}) + \mathbf{K}^T\mathbf{y}$. Defining an error variable $\mathbf{e} = \mathbf{y} - \mathbf{x}$ and taking the difference between Eq. (28) and Eq. (27) we obtain

$$d\mathbf{e}/dt = \mathbf{A}\mathbf{e} - \mathbf{C}\mathbf{K}^T\mathbf{e} = \mathbf{A}\mathbf{e} + \mathbf{C}p, \tag{29}$$

where $p = -\mathbf{K}^T\mathbf{e}$ plays the role of a state feedback. Note that Eq. (29) is in the standard form of linear feedback control theory and is the continuous counterpart of Eq. (5).

According to the standard pole placement procedure,[30] if the following controllability matrix

$$\mathbf{M} = [\mathbf{C}|\mathbf{A}\mathbf{C}|\mathbf{A}^2\mathbf{C}|\cdots|\mathbf{A}^{n-1}\mathbf{C}],$$

has full rank, then the eigenvalues of the matrix $\mathbf{A} - \mathbf{C}\mathbf{K}^T$ can be placed anywhere in the complex plane by choosing the proper **K**. In particular, those **K** vectors that lead to eigenvalues of negative real parts will make $\mathbf{e} = \mathbf{0}$ a globally attracting fixed point for Eq. (29). This means that the scalar signal constructed using such vectors will result in synchronization between the sender and the receiver.

If rank($\mathbf{M}$) $= q < m$ then there are $m - q$ eigenvalues that cannot be modified by the choice of $\mathbf{K}$. The condition for synchronization now is that the real parts of these uncontrollable eigenvalues be negative.

Many examples studied in the context of chaos synchronization in the literature have the same form of Eq. (27). The hyperchaotic Rössler equation considered earlier is an example. Specifically, for this example, we have

$$\mathbf{A} = \begin{pmatrix} 0 & -1 & -1 & 0 \\ 1 & 0.25 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -0.5 & 0.05 \end{pmatrix},$$

$\mathbf{C} = (0,0,1,0)^T$ and $\mathbf{D} = (0,0,3,0)^T$. The controllability matrix in this case is full rank. It is shown in Ref. 28 that by choosing $\mathbf{K} = (-3.3712, -0.9561, 4.3000, -5.8126)^T$ the eigenvalues of $\mathbf{A} - \mathbf{C}\mathbf{K}^T$ are placed exactly at -1.

In summary, by considering a special form of sender equations, the question of synchronizing hyperchaotic systems by a scalar signal is reduced to the study of the well developed linear control theory. In the more general case, however, it seems the method given by Peng *et al.*[26] is still a reasonable approach, although finding the region in the parameter space where the systems synchronize remains a challenging task.

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