ELSEVIER

Neurocomputing 32–33 (2000) 585–589

NEUROCOMPUTING

www.elsevier.com/locate/neucom

Stability constraints for oscillatory neural networks

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Accepted 13 January 2000

Abstract

The stability of the equilibrium point (background activity) of oscillatory neural networks is an important property for computational applications that explore the switching between background activity and oscillatory states. Here we consider a general approach to this problem for networks of arbitrary size. For symmetric coupling, often the case in associative learning algorithms, we derive the stability constraints and establish explicit results for the coupling strengths to satisfy in order that the equilibrium state is stable.

Keywords: Oscillatory networks; Cortical networks; Stability analysis; Background activity

1. Introduction

Computational properties of oscillatory networks have been a focus of research partly because their dynamics resemble the behavior often observed in the cortex and other brain regions. Many studies explore the property that these networks can switch between a background activity state and oscillatory states as the level of input varies [5,6,9]. Important in those applications is the fact that, once the input pattern has been removed, the network returns to its background or equilibrium state. This allows a natural form of resetting, making the network ready for the next computational cycle. To perform the requisite computations, the network is often subjected to associative learning, which leads to changes of the coupling strengths between its units. It becomes then fundamental to establish the range of variation of the coupling

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0925-2312/00/$ - see front matter
PII: S0925-2312(00)00215-0

values that preserve the stability of the background state. We investigate this stability problem by employing a general approach that can enable us to derive explicit stability conditions for the coupling strengths. Our main result states that the stability of the equilibrium state will be preserved if the individual coupling is confined in an interval that can be derived following a general procedure for a given system.

2. Network model

The basic unit in our network is a neuronal population consisting of either excitatory cells or inhibitory cells [2,8] that are organized in cortical columns. To facilitate our analysis, we only consider mutually excitatory interactions between columns. The equations for the N-column model read:

$$\frac{\mathrm{d}^2 x_n}{\mathrm{d}t^2} + (a+b)\frac{\mathrm{d}x_n}{\mathrm{d}t} + abx_n = -k_{\mathrm{ei}}Q(y_n, Q_{\mathrm{m}}) + \frac{1}{N-1}\sum_{p=1}^{N} k_{np}Q(x_p, Q_{\mathrm{m}}) + I_n,$$

$$\frac{\mathrm{d}^2 y_n}{\mathrm{d}t^2} + (a+b)\frac{\mathrm{d}y_n}{\mathrm{d}t} + aby_n = k_{\mathrm{ie}}Q(x_n, Q_{\mathrm{m}}). \tag{2.1}$$

Here x and y represent the local field potentials of the excitatory and inhibitory populations, respectively, $k_{\mathrm{ie}} > 0$ gives the coupling gain from the excitatory (x) to the inhibitory (y) population and $k_{\mathrm{ei}} > 0$ the strength of the reciprocal coupling, and $Q(x, Q_{\mathrm{m}})$ is a sigmoid function [1,3], representing pulse densities converted from x controlled by a modulatory parameter Q_{m}, with $Q(0, Q_{\mathrm{m}}) = 0$ and $Q'(0, Q_{\mathrm{m}}) = 1$. For simplicity, we have assumed that the inhibitory and excitatory populations have identical rate constants a and b. The columns are indexed by $n = 1, \ldots, N$ and the coupling strength k_{np} is the gain from the excitatory population of column p to the excitatory population of column n, with $k_{np} = 0$ for $n = p$.

3. Stability of the equilibrium point for N coupled columns

From (2.1), it is easy to verify that the origin, when the input is zero, is a fixed point which will be assumed here to be the background or equilibrium state. Below we find the constraints for the origin to be stable. For a single column, the application of the Routh–Hurwitz criterion to the linearized equation gives the following stability condition $k_{\mathrm{ie}}k_{\mathrm{ei}} < ab(a+b)^2$ which we assume to be always satisfied. For the N symmetrically coupled columns our main result can be stated as follows: there exists a $K^{\max}(a, b, k_{\mathrm{ie}}, k_{\mathrm{ei}})$, which is a function of the intracolumnar parameters a, b, k_{ie}, and k_{ei} such that the satisfaction of the constraint

$$0 \le k_{np} < K^{\max}(a, b, k_{\mathrm{ie}}, k_{\mathrm{ei}}) \tag{3.1}$$

ensures that the equilibrium point is stable. We establish this result in two steps that include the procedure for obtaining the $K^{\max}$ for a given system.

Step 1. The linearization of Eq. (2.1) can be written as: $\mathrm{d}\boldsymbol{S}/\mathrm{d}t = \boldsymbol{AS} + \boldsymbol{KSC}$ where $\boldsymbol{S}$ is a 4xN matrix containing the state variables of the model and $\boldsymbol{A}$ is the linearization matrix around the origin for a single column, i.e.,

$$\boldsymbol{S} \equiv \begin{bmatrix} x_1 & x_2 & \cdots & x_N \\ \dfrac{\mathrm{d}x_1}{\mathrm{d}t} & \dfrac{\mathrm{d}x_2}{\mathrm{d}t} & \cdots & \dfrac{\mathrm{d}x_N}{\mathrm{d}t} \\ y_1 & y_2 & \cdots & y_N \\ \dfrac{\mathrm{d}y_1}{\mathrm{d}t} & \dfrac{\mathrm{d}y_2}{\mathrm{d}t} & \cdots & \dfrac{\mathrm{d}y_N}{\mathrm{d}t} \end{bmatrix}$$

and

$$\boldsymbol{A} \equiv \begin{bmatrix} 0 & 1 & 0 & 0 \\ -ab & -(a+b) & -k_{\mathrm{ei}} & 0 \\ 0 & 0 & 0 & 1 \\ k_{\mathrm{ie}} & 0 & -ab & -(a+b) \end{bmatrix}.$$

The 4×4 matrix $\boldsymbol{K}$, whose only nonzero entry is $k_{21} = 1$, specifies the connectivity pattern and $\boldsymbol{C}$ is a $N \times N$ matrix containing the coupling values between the columns, i.e.,

$$C_{np} = \frac{k_{np}}{N-1}.$$

The above linear system can be rewritten as: $\mathrm{d}\boldsymbol{S}/\mathrm{d}t = \boldsymbol{AS} + \boldsymbol{KS}[\boldsymbol{E}\Lambda\boldsymbol{E}^{-1}]$, where $\boldsymbol{E}$ and Λ are the eigenvector and eigenvalue matrices of $\boldsymbol{C}$, respectively. Let $\boldsymbol{e}(n)$ be one of the eigenvectors of $\boldsymbol{E}$ and $\lambda(n)$ its associated eigenvalue. Introducing a new vector $\boldsymbol{u}(n) = \boldsymbol{Se}(n)$ we have for each decoupled eigenmode

$$\frac{\mathrm{d}\boldsymbol{u}(n)}{\mathrm{d}t} = [\boldsymbol{A} + \boldsymbol{K}\lambda(n)]\boldsymbol{u}(n). \tag{3.2}$$

The characteristic equation of $\boldsymbol{A} + \boldsymbol{K}\lambda(n)$ is $(r+a)^2(r+b)^2 - \lambda(n)[r^2 + (a+b)r + ab] + k_{\mathrm{ie}}k_{\mathrm{ei}} = 0$. Applications of the Routh–Hurwitz criterion yield a set of inequalities for $\lambda(n)$. Examination of these inequalities reveals that there is a $K^{\max}(a, b, k_{\mathrm{ie}}, k_{\mathrm{ei}})$ such that for $\lambda(n) < K^{\max}(a, b, k_{\mathrm{ie}}, k_{\mathrm{ei}})$ the eigenmode is stable. Let $\lambda^{\max}$ denote the largest eigenvalue of $\boldsymbol{C}$. Thus, the stability constraint for the entire system becomes

$$\lambda^{\max} < K^{\max}(a, b, k_{\mathrm{ie}}, k_{\mathrm{ei}}) \tag{3.3}$$

We note that once the values of a, b, k_{ie}, and k_{ei} are given the value of $K^{\max}(a, b, k_{\mathrm{ie}}, k_{\mathrm{ei}})$ is easily determined from the Routh–Hurwitz inequalities.

Step 2. To obtain the stability constraint on the coupling strength k_{np} we apply results from the theory of nonnegative matrices [4]. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two $N \times N$ matrices and $0 \le \boldsymbol{A} < \boldsymbol{B}$ with the inequality defined in terms of entry wise comparison. Then it can be shown that $\lambda^{\max}(\boldsymbol{A}) < \lambda^{\max}(\boldsymbol{B})$. Let $\boldsymbol{B}$ be the coupling matrix with all the

entries equal to

$$C_{np} = \frac{K^{\max}(a, b, k_{\mathrm{ie}}, k_{\mathrm{ei}})}{N-1}.$$

It is clear that $\lambda^{\max}(\boldsymbol{B}) = K^{\max}(a, b, k_{\mathrm{ie}}, k_{\mathrm{ei}})$. If $\boldsymbol{A}$ is any coupling matrix with $0 \le k_{np} < K^{\max}(a, b, k_{\mathrm{ie}}, k_{\mathrm{ei}})$, then $\lambda^{\max}(\boldsymbol{A}) < K^{\max}(a, b, k_{\mathrm{ie}}, k_{\mathrm{ei}})$. We thus establish the main result, the stability constraint (3.1), stated at the beginning of this section.

4. Conclusions

A principled way of establishing stability conditions for large neural networks was presented. The approach was applied to derive explicit stability constraints for the coupling strengths in the case of an oscillatory network where each unit is described by a second-order nonlinear ODE. The extension of the same approach to the case of asymmetric coupling and to more general types of network units is developed in [7].

Acknowledgements

This work was supported by a fellowship from Brazilian government research agency CNPq (n. 200255/96-7), Office of Naval Research (grant 00014-99-1-0062) and National Institute of Mental Health (grant 58190-02).

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