X. Cross Spectral Analysis
Cross Spectrum

We have already dealt with the crosscovariance function (ccvf):

\[ \hat{C}_{XY}(k) = \frac{1}{N} \sum_{i=0}^{N-k} X(i)Y(i + k) \]
The Fourier transform of the ccvf is called the cross spectrum. It provides a statement of how common activity between two processes is distributed across frequency.

\[ \hat{Z}_{XY}(n) = \frac{1}{N} \sum_{k=1}^{N} \hat{C}_{XY}(k\Delta t)e^{-jn\omega_0 k\Delta t} \]

The cross spectrum is a complex-valued function. This is because, unlike the autocovariance function, which is even, the cosscovariance function is not symmetric around \( \tau = 0 \). This means that the sine terms of the exponential do not cancel as for the power spectrum.
The same procedures and problems apply to estimating cross spectra as were discussed for power spectra. The cross spectrum can be estimated from the ccvf or from the Fourier transforms of the two time series.
In the frequency domain, we compute the cross spectrum as:

\[
\hat{Z}_{XY}(n) = \hat{Z}_X(n)\hat{Z}_Y^*(n)
\]

\[
= \left[ \frac{A_X(n) - jB_X(n)}{2} \right] \left[ \frac{A_Y(n) + jB_Y(n)}{2} \right]
\]

\[
= \left[ \frac{A_X(n)A_Y(n) + B_X(n)B_Y(n)}{4} \right] - j\left[ \frac{B_X(n)A_Y(n) - B_Y(n)A_X(n)}{4} \right]
\]

\[
= \hat{C}_{o_{XY}}(n) - j\hat{Q}_{u_{XY}}(n)
\]

where \(\hat{C}_{o_{XY}}(n)\) is called the cospectrum and \(\hat{Q}_{u_{XY}}(n)\) is called the quadrature spectrum.
Then we define the cross-power as the sum of the real and imaginary parts squared:

$$|\hat{Z}_{XY}(n)|^2 = \hat{C}_{O_{XY}}(n)^2 + \hat{Q}_{u_{XY}}(n)^2$$
Coherence

The coherence function is a measure based upon the auto- and cross-spectral properties of two processes. Its definition is identical in form to the square of a correlation coefficient between the spectral components of the processes at a particular frequency $f$.

Note that two factors contribute to a significant value of the cross-spectrum at a particular frequency. One is a significant coupling between the two processes at that frequency. The other is a significant concentration of power at that frequency in one or both processes.
The coherence function, or squared coherence, is defined as:

\[
\hat{\gamma}_{XY}^2 = \frac{|\hat{Z}_{XY}(n)|^2}{|\hat{Z}_{XX}(n)|^2 |\hat{Z}_{YY}(n)|^2}
\]

where:

\[
|\hat{Z}_{XX}(n)|^2 = \frac{A_{XX}(n)^2 + B_{XX}(n)^2}{4},
\]

\[
|\hat{Z}_{YY}(n)|^2 = \frac{A_{YY}(n)^2 + B_{YY}(n)^2}{4}
\]
\[ \gamma_{XY}^2 = \frac{\left| \hat{Z}_{XY}(n) \right|^2}{\left| \hat{Z}_{XX}(n) \right|^2 \left| \hat{Z}_{YY}(n) \right|^2} \]

The cross-power in the numerator ranges from 0 to the product of the two autospectra in the denominator. In other words, the coherence is a normalization of the cross power spectrum by the product of the two autospectra.

The normalization of the coherence compensates for large values in the cross spectrum that result solely from large power, and thus isolates the influence of coupling.
If X and Y are identical processes, then their power spectra are equal to each other and to the cross spectrum. In this case, the coherence is 1 at all frequencies.

If X and Y are completely independent processes, then the cross spectrum is 0 for all frequencies and so is the coherence.

Between these two extremes lies a range of possible relationships between the two processes that can be measured by the coherence function.
X and Y may be closely related, but only over a limited range of frequencies. If, for example, two regions of the hippocampus are independent of each other for a large part of the time, but are both intermittently driven by septal input within the narrow theta frequency band. In this case, coherence between the time series of the two regions would be near unity over the theta band, and zero elsewhere.
Example 2

A similar situation would occur when comparing the left and right motor cortices. At rest or during unsynchronized movements, their EEGs might be largely independent. During synchronized bilateral movement, there could arise a component in each EEG at the frequency of the movement. The coherence of the EEGs during the synchronized movement would then be large over the frequencies of synchronization, and small elsewhere.
A meaningful estimate of coherence cannot be obtained directly from single samples of auto- and cross-spectra of the processes. Consider what would result from using these “raw” spectra:

\[ \frac{|Z_{XY}(n)|^2}{|Z_{XX}(n)|^2|Z_{YY}(n)|^2} = \frac{C_{OXY}(n)^2 + Q_{u_{XY}}(n)^2}{|Z_{XX}(n)|^2 |Z_{YY}(n)|^2} \]

\[ \frac{(A_xA_y)^2 + 2A_xA_yB_xB_y + (B_xB_y)^2 + (B_xA_y)^2 + (B_yA_x)^2 - 2A_xA_yB_xB_y}{16} \]

\[ = \frac{16}{(\frac{A_x^2 + B_x^2}{4})(\frac{A_y^2 + B_y^2}{4})} \]

\[ = \frac{A_x^2A_y^2 + B_x^2B_y^2 + A_y^2B_x^2 + A_x^2B_y^2}{A_x^2A_y^2 + B_x^2B_y^2 + A_y^2B_x^2 + A_x^2B_y^2} \]

\[ = 1 \]

The coherence function is thus an estimate that must be derived after first estimating the co-spectrum, quadrature spectrum, and the auto-spectra. The same bias-resolution and variance considerations apply as for the power spectrum.
Generally, small values of coherence are encountered in dealing with the EEG. Electrode sites usually do not have large correlations, unless they are near each other.

Example: Consider a simple model in which two processes are each composed of a common quasiperiodic signal process embedded in independent wide-band noise processes. Suppose the quasiperiodic signals are due to a common phenomenon so that they are closely related. The wide band noise processes, on the other hand, are due to random fluctuations that are unique to each process and so are unrelated.
The cross spectrum of the two processes would be relatively large in the frequency band of the shared, quasiperiodic signal and small at other frequencies, since the wide band noise processes are independent and not shared activity.
This can be expressed mathematically as:

\[ x(t) = n_1(t) + s(t) \]
\[ y(t) = n_2(t) + s(t) \]

The power spectral representation is:

\[ S_{xx}(n) = S_{n_1n_1}(n) + S_{ss}(n) \]
\[ S_{yy}(n) = S_{n_2n_2}(n) + S_{ss}(n) \]
\[ S_{xy}(n) = |Z_{xy}(n)|^2 \]
However, the cross power, which is the Fourier transform of the ccvf, solely depends on the signal since the only covariance between X and Y is that caused by the presence of the signal in both. Based on this assumption, the cross power is:

\[
|Z_{xy}(n)|^2 = C_{xy}(n)^2 + Q_{xy}(n)^2 \\
= \left( \frac{(A_x(n)A_y(n))^2 + (B_x(n)B_y(n))^2 + (A_y(n)B_x(n))^2 + (A_x(n)B_y(n))^2}{16} \right) \\
= \left( \frac{A_s(n)^2 + B_s(n)^2}{4} \right)^2 = S_{ss}(n)^2
\]
The coherence function is then:

\[
\gamma_{xy}^2(n) = \frac{S_{ss}(n)^2}{[S_{n1n1}(n) + S_{ss}(n)][S_{n2n2}(n) + S_{ss}(n)]}
\]

If we let \( n1 \) and \( n2 \) have identical spectra, then the coherence becomes:

\[
\gamma_{xy}^2(n) = \frac{S_{ss}(n)^2}{[S_{nn}(n) + S_{ss}(n)]^2}
\]

\[
= \frac{1}{[1 + \frac{S_{nn}(n)}{S_{ss}(n)}]^2}
\]
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\]

If SNR = 1, coherence = .25. If SNR = .5, coherence = .11.
Coherence estimates have been devised and their 95% confidence limits determined as a function of the number of segments used in Bartlett smoothing. The graph shows not only the 95% confidence range, but also the minimum coherence value needed to confirm any coherence with 95% confidence. Large estimation errors are likely when the coherence value is low.

Fig. 3.11. The 95% confidence intervals of the coherence function, plotted for the number of data segments used in smoothing. The discontinuities in the upper bounds reflect the change to a one-tailed interval when the lower confidence limit descends to 0. [Benignus, V. A., IEEE Trans. Audio Electroacoust. AU-17, 145 (1969).]
Relative Phase

The coherence does not include any phase information concerning the two processes. Rather, it considers their relationship only in terms of the power at a given frequency. The relative phase spectrum can be derived from the cross spectrum as:

$$\hat{\theta}_{XY}(n) = \arctan\left(-\frac{\hat{Qu}_{XY}(n)}{\hat{Co}_{XY}(n)}\right)$$

where the denominator is the real part (co-spectrum) of the cross spectrum, and the numerator is the imaginary part (quadrature spectrum).
If the processes X and Y are uncorrelated, then no particular relative phase relationship is to be expected at any frequency. Relative phase will be a random variable with 0 mean and uniformly distributed over the range -90 to +90 degrees. When there is a correlation between the two processes, the relative phase relation will show up as a preferred relative phase angle related to frequency.
The relative phase spectrum, like the coherence spectrum, is only useful as an estimate. Furthermore, it is only valid at frequencies where there is a significant coherence value. This is because the coherence is an indicator of the amount of dispersion around the mean relative phase of the distribution of phase values of the ensemble of data realizations. A significant coherence value indicates that the relative phase is concentrated around the mean. If it is not significant, then the relative phase distribution is not significantly concentrated, and the mean relative phase will not be meaningful.