VI. Power Spectra
Amplitude and Phase

We start with the Fourier series representation of $X(t)$ in real notation:

$$X(t) = \frac{A(0)}{2} + \sum_{n=1}^{\infty} \left[ A(n)\cos(n\omega_0 t) + B(n)\sin(n\omega_0 t) \right]$$
The waveform of the observed segment $T$ exactly determines the values of the Fourier coefficients:

$$A(n) = \frac{2}{T} \int_{0}^{T} X(t) \cos(n\omega_0 t) dt$$

and

$$B(n) = \frac{2}{T} \int_{0}^{T} X(t) \sin(n\omega_0 t) dt$$
As long as $X(t)$ is continuous, the complete Fourier series obtained in this manner will reconstruct the original waveform without error.

Once $A(n)$ and $B(n)$ have been determined in this way, they can be plotted as functions of frequency $(n\omega_0)$. These spectra are defined only for integer values of $n$. For this reason, they are called **discrete spectra** or **line spectra**.
A(n) and B(n) both refer to the same frequency; together they are equivalent to the complex spectrum:

\[ Z(n) = \frac{A(n) - jB(n)}{2} \]
Z(n) is a complex number.

The absolute value or amplitude of Z(n) is defined as:

\[ |Z(n)| = \sqrt{Z(n)Z(n)^*} = \sqrt{A(n)^2 + B(n)^2} \]

The amplitude of Z(n) displayed as a function of n (or frequency) is referred to as the amplitude spectrum of x(t).
Consider the following trigonometric expression:

\[ A \cos x + B \sin x = \sqrt{A^2 + B^2} \left( \frac{A}{\sqrt{A^2 + B^2}} \cos x + \frac{B}{\sqrt{A^2 + B^2}} \sin x \right) \]

Letting: \( \cos \theta = \frac{A}{\sqrt{A^2 + B^2}} \) and \( \sin \theta = \frac{B}{\sqrt{A^2 + B^2}} \)

and letting \( x = n\omega_0 \),

and using the following trigonometric expression:

\[ \cos x \cos \theta + \sin x \sin \theta = \cos (x - \theta) \]
We find that:

\[ A(n) \cos(n\omega_0 t) + B(n) \sin(n\omega_0 t) = \sqrt{A(n)^2 + B(n)^2} \cos[n\omega_0 t - \theta(n)] \]

Notice that the amplitude is \((A^2 + B^2)^{1/2}\)
rather than \((A^2 + B^2)^{1/2} / 2\).

This is because the expression is written in real notation, and the amplitude at \(n\) has equal contributions from the complex components at \(\mathbf{–n}\) and \(n\).
Notice that $\tan(\theta) = (B/A)$.

From this, it follows that the angle $\theta$, called the phase, is:

$$\theta(n) = \arctan\left(\frac{B(n)}{A(n)}\right)$$

When displayed as a function of $n$ (or frequency), $\theta$ yields the phase spectrum of $X(t)$. 
Power

$Z(n)|^2$ is referred to as power:

$$
|Z(n)|^2 = \frac{A(n)^2 + B(n)^2}{4}
$$

When displayed as a function of n or frequency, it becomes the power spectrum of $X(t)$.

The term power comes from the field of electrical engineering, where power dissipated in an electrical circuit is proportional to the mean square voltage applied.
The total power of a time series can be determined either directly from the time series or from its spectral components. In the time domain, power is defined as the mean square (ms) value:

$$ms = \frac{1}{T} \int_{0}^{T} X(t)^2 \, dt$$

or its root:

$$rms = \sqrt{\frac{1}{T} \int_{0}^{T} X(t)^2 \, dt}$$
Parseval's Theorem

Parseval's theorem relates the mean square value and the sum of spectral powers as:

\[
\frac{1}{T} \int_{0}^{T} X(t)^2 \, dt = \sum_{n=-\infty}^{\infty} |Z(n)|^2
\]

Thus the total power of \( X(t) \) can be determined in the time domain as the mean square value, or in the frequency domain as the sum of spectral powers.
• The complex spectrum is symmetric around $\omega = 0$, so the complex components at $n$ and $-n$ are equal.

• The total power at any frequency (in real notation) is equally divided between complex components at $n\omega_0$ and $-n\omega_0$.

$$|Z(n)|^2 = |Z(-n)|^2 \text{ for } 0 \leq n \leq \infty$$

and

$$\text{Total Power at } n = |Z(n)|^2 + |Z(-n)|^2 = 2|Z(n)|^2 \text{ for } 1 \leq n \leq \infty$$

Nb: $n$ is both negative and positive in complex notation; it is only non-negative in real notation.
An Example

Now we consider a simple example in which we determine the total power for a cosine function with amplitude of 1 at an arbitrary frequency $k\omega_0$.

For $X(t)=\cos(k\omega_0 t)$:
(1) Derive the values for $A(n)$ and $B(n)$ for all values of $n$.
(2) Using $A(n)$ and $B(n)$, compute the total power of $X(t)$. 


First, solve for $A(n)$:

$$A(n) = \frac{2}{T} \int_0^T X(t) \cos(n \omega_0 t) \, dt = \frac{2}{T} \int_0^T \cos(k \omega_0 t) \cos(n \omega_0 t) \, dt$$

In the case where $k \neq n$, we can use the following integral form:

$$\int \cos(ax) \cos(bx) \, dx = \frac{\sin[(a-b)x]}{2(a-b)} + \frac{\sin[(a+b)x]}{2(a+b)}$$

with $a = k \omega_0$, $b = n \omega_0$, $x = t$ (provided $a \neq b$)
It follows that:

\[
A(n) = \frac{2}{T} \int_0^T \cos(k \omega_0 t) \cos(n \omega_0 t) \, dt = \frac{\sin[(k-n) \omega_0 t]}{2(k-n) \omega_0} + \frac{\sin[(k+n) \omega_0 t]}{2(k+n) \omega_0} \bigg|_0^T
\]

\[
= \frac{\sin[(k-n)2\pi]}{2(k-n) \omega_0} + \frac{\sin[(k+n)2\pi]}{2(k+n) \omega_0} - \frac{\sin[0]}{2(k-n) \omega_0} - \frac{\sin[0]}{2(k+n) \omega_0}
\]

\[
= 0 + 0 - 0 - 0 = 0
\]

Conclusion: \( A(n) = 0 \) for all \( n \) where \( k \neq n \).
Now consider the case where \( k = n \).
We can use the following integral form:

\[
\int \cos^2 ax \, dx = \frac{x}{2} + \frac{\sin 2ax}{4a}
\]

Again, we let \( a = k\omega_0, \, b = n\omega_0, \, x = t \). Then:

\[
A(k) = \frac{2}{T} \int_0^T \cos(\kappa t)^2 \, dt = \frac{2}{T} \left[ \frac{t}{2} + \frac{\sin(2\kappa_0 t)}{4\kappa_0} \right]_0^T
\]

\[
= \frac{2}{T} \left[ \frac{T}{2} + \frac{\sin(2\kappa_2\pi)}{4\kappa_0} - \frac{0}{2} - \frac{\sin(2\kappa_0)}{4\kappa_0} \right]
\]

\[
= \frac{2}{T} \left[ \frac{T}{2} + \frac{\sin(2\kappa_2\pi)}{4\kappa_0} - \frac{\sin(0)}{4\kappa_0} \right] = \frac{2}{T} \frac{T}{2} = 1
\]

Conclusion: \( A(n) = 1 \) when \( k = n \).
Next, solve for $B(n)$:

$$B(n) = \frac{2}{T} \int_0^T X(t) \sin(n\omega_0 t) \, dt = \frac{2}{T} \int_0^T \cos(k\omega_0 t) \sin(n\omega_0 t) \, dt$$

In the case where $k \neq n$, we can use the following integral form:

$$\int \cos ax \sin bx \, dx = \frac{\cos[(a + b) x]}{2(a + b)} - \frac{\cos[(b - a) x]}{2(b - a)}$$

with $a = k\omega_0$, $b = n\omega_0$, $x = t$ (provided $a \neq b$)
It follows that:

\[ B(n) = \frac{2}{T} \int_0^T \cos(k \omega_0 t) \sin(n \omega_0 t) \, dt = \frac{\cos[(k+n) \omega_0 t]}{2(k+n) \omega_0} - \frac{\cos[(n-k) \omega_0 t]}{2(n-k) \omega_0} \bigg|_0^T \]

\[ = \frac{\cos[(k+n) 2\pi]}{2(k+n) \omega_0} - \frac{\cos[(n-k) 2\pi]}{2(n-k) \omega_0} - \frac{\cos[(k+n) 0]}{2(k+n) \omega_0} + \frac{\cos[(n-k) 0]}{2(n-k) \omega_0} \]

\[ = \frac{1}{2(k+n) \omega_0} - \frac{1}{2(n-k) \omega_0} - \frac{1}{2(k+n) \omega_0} + \frac{1}{2(n-k) \omega_0} = 0 \]

Conclusion: \( B(n) = 0 \) for all \( n \) where \( k \neq n \).
Now consider the case where \( k = \pm n \).

We can use the following integral form:

\[
\int \sin ax \cos ax \, dx = -\frac{\cos 2ax}{4a}
\]

Again, we let \( a = k\omega_0, \ b = n\omega_0, \ x = t \). Then:

\[
B(k) = \frac{2}{T} \int_0^T \cos(k\omega_0 t) \sin(k\omega_0 t) \, dt = -\frac{\cos(2k\omega_0 t)}{4k\omega_0} \bigg|_0^T
\]

\[
= -\frac{\cos(2k2\pi)}{4k\omega_0} + \frac{\cos(2k0)}{4k\omega_0} = -\frac{1}{4k\omega_0} + \frac{1}{4k\omega_0} = 0
\]

Conclusion: \( B(n) = 0 \) when \( k = n \).
<table>
<thead>
<tr>
<th>k</th>
<th>A(n)</th>
<th>B(n)</th>
</tr>
</thead>
<tbody>
<tr>
<td>≠ n</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>n</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>
To solve for total power:

\[
Total\ Power = \sum_{n=-\infty}^{\infty} |Z(n)|^2 = |Z(k)|^2 + |Z(-k)|^2 = 2 \frac{[A^2 + B^2]}{4} = \frac{1}{2}
\]
More Examples

(1) Two time series may be very different and yet have identical power spectra. They would necessarily have different phase spectra.
(2) Reconstruction of a time series using only the components having largest power yields a smoothed version of the time series.
(3) The power spectrum is useful for distinguishing between time series that may look alike but are generated by very different processes, e.g. beat (2 close frequency components) vs amplitude modulation (1 component that changes amplitude over time).
(4) The power spectrum of the EEG is useful for distinguishing among different brain states, e.g. delta for deep sleep, alpha for resting awake, and broad-band for focused awake.
The power spectrum of a time series only displays the average power over the length of the time series. Two very different time functions, e.g. “low amplitude with persistent rhythmic activity” vs “irregularly occurring bursts of high amplitude rhythmic activity”, will have the same power spectrum if their average power is the same.
Relation Between Power Spectrum and Autocovariance

Remember that for a signal of length $T$, the estimate of the autocovariance function for an aperiodic function is defined as:

$$\hat{C}_{xx}(\tau) = \frac{1}{T} \int_{0}^{T-|\tau|} X(t)X(t+\tau)\,dt$$
We can now express $X(t)$ and $X(t+\tau)$ as their Fourier series representations.

\[
X(t) = \sum_{n=-\infty}^{\infty} Z(n)e^{jn\omega_0 t}
\]

\[
X(t + \tau) = \sum_{m=-\infty}^{\infty} Z(m)e^{jm\omega_0 (t+\tau)}
\]
We now substitute these Fourier series representations in the autocovariance equation:

\[
\hat{C}_{xx}(\tau) = \frac{1}{T} \int_0^{T-|\tau|} \left( \sum_{n=-\infty}^{\infty} Z(n)e^{jn\omega_0 t} \right) \left( \sum_{m=-\infty}^{\infty} Z(m)e^{jm\omega_0 (t+\tau)} \right) dt
\]

\[
= \sum_{n,m=-\infty}^{\infty} Z(n)Z(m) \left( \frac{1}{T} \right) \int_0^{T-|\tau|} e^{jn\omega_0 t} e^{jm\omega_0 t} e^{jm\omega_0 \tau} dt
\]

Since \( \tau \) is constant with respect to the integral:

\[
\hat{C}_{xx}(\tau) = \sum_{n,m=-\infty}^{\infty} Z(n)Z(m)e^{jm\omega_0 \tau} \left[ \left( \frac{1}{T} \right) \int_0^{T-|\tau|} e^{j(n+m)\omega_0 t} dt \right]
\]
Next, consider the expression within the [] in this equation. First of all, if \((n+m) \neq 0\), then the integral of the exponential in the [] is approximately equal to 0. It is exactly equal to 0 when \(\tau = 0\) since \((n+m)\) is an integer multiplier of \(\omega_0\), and each sine and cosine component thus has an integer number of cycles in interval \(T\). For this reason, the integral over one period of a sine or cosine wave has positive and negative values that exactly cancel. When the interval is reduced to length \(T-\tau\), the positive and negative values of the cosine terms will not exactly cancel, but the residual integral will still be close to 0. The result is that we need not consider values of the integral in the [] for which \((n+m) \neq 0\) since the contribution of these terms to the autocovariance function is negligible.

Next, consider the situation where \((n+m) = 0\). In this case, the exponential inside the integral in [] is equal to 1, and the integral will equal the length over which the integration is performed. When \(\tau = 0\), this length is \(T\), and multiplication of the integral by \(1/T\) causes the term in [] to exactly equal 1. When the length is \(T-\tau\), division by \(1/T\) yields terms that are negligibly less than 1. The result is that value of the integral in the [] needs only be considered in the case where \((n+m) = 0\), and the term in [] is approximately 1 in that case.
Therefore, we conclude that:

\[ \hat{C}_{XX}(\tau) = \sum_{n,m=-\infty}^{\infty} Z(n)Z(m)e^{jm\omega_0 \tau} \]

Since \( n + m = 0 \), \( m = -n \). We thus substitute \(-n\) for \( m \), giving:

\[ \hat{C}_{XX}(\tau) = \sum_{n=-\infty}^{\infty} Z(n)Z(-n)e^{-jn\omega_0 \tau} \]
We have already seen that $Z(n)$ and $Z(-n)$ are complex conjugates:

$$Z(n) = \frac{A(n) - jB(n)}{2}$$

$$Z(-n) = \frac{A(n) + jB(n)}{2}$$
Therefore, the product of $Z(n)$ and $Z(-n)$ is the power. Thus:

$$Z(n)Z(-n) = \frac{A(n)^2 + B(n)^2}{4} = |Z(n)|^2$$

$$\hat{C}_{XX}(\tau) = \sum_{n=-\infty}^{\infty} |Z(n)|^2 e^{-jn\omega_0 \tau}$$
Note that:

\(-\infty < n < \infty\)

and

\[ |Z(n)|^2 = |Z(-n)|^2 \]

Therefore:

\[
\hat{C}_{XX}(\tau) = \sum_{n=-\infty}^{\infty} |Z(n)|^2 e^{jn\omega_0 \tau}
\]

This relation shows that the autocovariance function can be expressed as a complex Fourier series whose coefficients are the power values.
This means that the autocovariance and power spectrum form a Fourier transform pair, so that:

\[ |Z(n)|^2 = \frac{1}{T} \int_0^T \hat{C}_{XX}(\tau)e^{-jn\omega_0 \tau} d\tau \]

and

\[ \hat{C}_{XX}(\tau) = \sum_{n=-\infty}^{\infty} |Z(n)|^2 e^{jn\omega_0 \tau} \]
Note that the Fourier transform of the autocovariance function is a real-valued function. This is because the products of all the sine terms of the exponential and $C_{xx}(\tau)$ are zero. This results from the fact that $C_{xx}(\tau)$ is symmetric around $\tau = 0$, and the sine functions are all anti-symmetric. This confirms what we already know, namely that the power spectrum is a real function. It is also why the Fourier transform of $C_{xx}(\tau)$ is called Fourier cosine transformation.

Note also that, since real and imaginary terms have been combined in the power, it is not possible to determine phase from the Fourier representation of the autocovariance. Since the phase information is not available, it is not possible to deduce the original waveform of the time series from the autocovariance. Any number of time series with different phase structure could produce the same autocovariance function.
Finally, remember that $C_{xx}(0)$ is the variance of $X(t)$. Assuming that the mean of $X(t)$ is 0, the variance is equivalent to the mean square value. Then:

$$C_{xx}(0) = \frac{1}{T} \int_0^T X(t)^2 \, dt = \sum_{n=-\infty}^{\infty} |Z(n)|^2 e^0 = \sum_{n=-\infty}^{\infty} |Z(n)|^2$$

This proves Parseval’s theorem.