XII. Bandwidth Limited Time Series
To summarize the discussion up to this point:

(1) In the general case of the aperiodic time series, which is infinite in time and frequency, both the time series and the Fourier series are continuous functions.

(2) When we consider a time series of finite length \( T \), the Fourier series becomes discrete at integer multiples of the fundamental frequency, which is inversely related to \( T \).

(3) However, at this point the time series is still continuous and the frequency ranges from negative to positive infinity (or from 0 to positive infinity using real notation). The frequencies that can be resolved are the fundamental frequency and all integer multiples of it.
Consider now the case where, in addition to being of finite length, the time series is also discrete rather than continuous. This is the situation that results when a continuous analog signal is digitized at a fixed sampling interval $\Delta t$. The sampled time series is now referred to as $X(k \Delta t)$, where $k$ indexes the interval.

Discrete temporal sampling has the effect of limiting the highest frequency that can be resolved.
The minimum number of time points necessary to specify one cycle at any frequency is 3, one at the beginning, one at the middle, and one at the end of the cycle.
Given an arbitrary sampling interval $\Delta t$:

(a) 3 time points is equivalent to $2\Delta t$.

(b) 1 cycle in $2\Delta t$ is the minimally defined frequency that is uniquely determined by this sampling interval.

(c) this frequency is the maximum frequency that can be resolved with this sampling interval. (No higher frequency can be specified with 3 time points.)
Nyquist Frequency

The maximum frequency that can be resolved is called the *Nyquist frequency*. It represents 1 cycle in $2\Delta t$.

$$F = \frac{1}{2\Delta t} \text{cycles/sec (Hz)}$$

$$= \frac{2\pi}{2\Delta t} \text{radians/sec}$$

There being a maximum resolvable frequency means that the spectrum of a sampled time series is *bandwidth (or band) limited* between 0 and F.
When a continuous time series is sampled, the sampling rate (S), which is the number of samples per unit of time, equals $1/\Delta t$. Therefore, the Nyquist frequency ($F$) is:

$$F = \frac{1}{2\Delta t} = \frac{1}{2 \frac{1}{S}} = \frac{S}{2}$$
We now define \( N \) as the number of sampled intervals within the observation period \( T \).

\[ N = \frac{T}{\Delta t} \quad (\text{and} \quad \Delta t = \frac{T}{N} \quad \text{and} \quad T = N \Delta t). \]

Remember that the index \( n \) represents the integer multiplier of the fundamental frequency in the Fourier series representation. The Nyquist frequency is itself part of that representation, so there must be a value of \( n \) corresponding to the Nyquist frequency. That value of \( n \) is \( N/2 \):

\[
F = \frac{2\pi}{2\Delta t} = \frac{2\pi N}{2T} = \left( \frac{N}{2} \right) \left( \frac{2\pi}{T} \right)
\]

Since \( \omega_0 = \frac{2\pi}{T} \):

\[
F = \left( \frac{N}{2} \right) \omega_0
\]
We can now write the Fourier series representation of the discrete time series $X(k\Delta t)$ (in real notation):

$$X(k\Delta t) = \frac{A(0)}{2} + \sum_{n=1}^{N/2} \left[ A(n) \cos(n\omega_0 k\Delta t) + B(n) \sin(n\omega_0 k\Delta t) \right]$$

We see that $n$ ranges from 1 to $N/2$. Note that there are $N/2$ Fourier components in addition to $A(0)$. Since each component has 2 coefficients, the total number of coefficients needed to represent $X(k\Delta t)$ would appear to be $N+1$. However, it turns out that $B(N/2)=B(0)=0$. Therefore, a total of $N$ coefficients is needed to represent $X(k\Delta t)$. 
Bandwidth-limited DFT

In complex notation, the bandwidth-limited versions of the DFT and the inverse DFT are:

\[ Z(n) = \frac{1}{N} \sum_{k=1}^{N} X(k \Delta t) e^{-j n \omega_0 k \Delta t} \]

and

\[ X(k \Delta t) = \sum_{n=-\frac{N}{2}}^{\frac{N}{2}} Z(n) e^{j n \omega_0 k \Delta t} \]
If the coefficients of the Fourier series for the continuous time series $X(t)$ are all 0 for frequencies higher than the Nyquist frequency, then sampling at $\Delta t$ is guaranteed to represent all the information in the time series.

In other words, if all the power of $X(t)$ is contained in frequency components at the Nyquist frequency or lower, then sampling at $\Delta t$ will be adequate to evaluate the Fourier coefficients uniquely.

That is, the continuous time series $X(t)$ can be completely reconstructed by the sampled time values, provided it is band limited by the Nyquist frequency.

In short, all is well if the time series bandwidth is less than the Nyquist frequency.
When the time series bandwidth extends beyond the Nyquist frequency, a number of difficulties result. The Fourier coefficients may be computed in the usual way, but they will suffer an error whose size depends on the amount by which the bandwidth exceeds the Nyquist frequency. The greater this excess, the greater the errors and the less adequate are the samples as a representation of the time series. These errors may result in serious misinterpretations if unrecognized. The onus is upon the analyst to ensure that sampling is adequate for the bandwidth of the time series being analyzed.
One common way this is handled is to by the use of low-pass analog filtering prior to A-D conversion. The bandwidth of the data must be limited to the Nyquist frequency PRIOR to sampling in order to prevent these errors.

Example: If one decides to sample at 200 samples/sec, then $\Delta t$ is 5 msec and the Nyquist frequency is 100 c/sec. Then, to prevent errors the analog time series must be lowpass filtered to remove any appreciable power above 100 c/sec.
Aliasing

The question now is how spectral analysis can be falsified by sampling when there is power at frequencies above the Nyquist frequency. Let us consider the case where we sample a continuous time series $X(t)$, assuming that it is bandwidth limited, when in fact it is not. The true DFT of the time series is:

$$Z(n) = \frac{1}{T} \int_{0}^{T} x(t) e^{-jn\omega_0 t} dt$$
We apply the bandwidth-limited version of the DFT to the sampled time series to get the apparent Fourier coefficients:

\[
Z(n) = \frac{1}{N} \sum_{k=1}^{N} X(k\Delta t)e^{-jn\omega_0 k\Delta t}
\]

where \( T = N\Delta t \).
By sampling, we have lost time points that contained the fine structure of the time series. So, in the frequency domain, components above the Nyquist frequency can no longer be represented properly.

What happens to these higher frequency components? To answer this question, we consider the true Fourier series representation, which is given by:

$$X(t) = \sum_{m=\infty}^{\infty} Z(m) e^{jm\omega_0 t}$$

The integer $m$ represents all the frequency components in the Fourier series representation of $X(t)$. We want to see how these frequency components distribute when we naively perform the bandwidth-limited DFT (i.e. incorrectly assuming that there is no power above the Nyquist frequency).
In essence, we are using the bandwidth limited DFT (with frequencies \( n \leq N/2 \)):

\[
Z(n) = \frac{1}{N} \sum_{k=1}^{N} X(k\Delta t)e^{-jn\omega_0k\Delta t}
\]

to represent a time series with frequency components (\( m \)) that cannot properly be represented because they are above the Nyquist frequency:

\[
X(t) = \sum_{m=-\infty}^{\infty} Z(m)e^{jm\omega_0t}
\]

To understand the errors this causes, we must explore the relation between \( n \) (frequencies within the Nyquist range) and \( m \) (all frequencies in \( X(t) \)).
We substitute the second equation into the first, giving:

\[ Z(n) = \frac{1}{N} \sum_{k=1}^{N} \left[ \sum_{m=-\infty}^{\infty} Z(m) e^{im\omega_0 k \Delta t} e^{-jn\omega_0 k \Delta t} \right] = \frac{1}{N} \sum_{k=1}^{N} \left[ \sum_{m=-\infty}^{\infty} Z(m) e^{j(m-n)\omega_0 k \Delta t} \right] \]

Rearranging, we get:

\[ Z(n) = \frac{1}{N} \sum_{k=1}^{N} \left[ \sum_{m=-\infty}^{\infty} Z(m) e^{j(m-n)\omega_0 k \Delta t} \right] = \sum_{m=-\infty}^{\infty} Z(m) \left\{ \left( \frac{1}{N} \right) \sum_{k=1}^{N} e^{j(m-n)\omega_0 k \Delta t} \right\} \]
We next need to resolve the expression in curly brackets.

(a) First note that the frequencies involved are \((m-n)\omega_0\), where \(m\) ranges from \(-\infty\) to \(\infty\).

(b) Next consider that for any frequency of the form \((m-n)\omega_0\), an integer number of periods exists in the time interval from \(k = 1\) to \(N\).

(c) Since the sum of a cosine or sine wave over an integer number of periods equals 0, the sum of the exponential term \(\exp[j\omega_0(m-n)k\Delta t]\) over the time interval equals 0. Then, the expression in curly brackets also equals 0.

(d) However, there is an exception to this general conclusion. In the special case when \((m-n) = 0\), \(e^0 = 1\), and the sum over the time interval equals \(N\). Then, the expression in curly brackets equals 1.
(e) In fact, there is also an exception to the general conclusion if \((m-n)\) is a non-zero integer multiple of \(N\), i.e. \((m-n)=iN\), where \(i\) is a non-zero integer multiplier of \(N\).

(f) If \((m-n) = iN\), with \(i \neq 0\), then the exponential term \(\exp[j\omega_0 iN k \Delta t]\) reduces to \(\exp(j2\pi ik)\). [Here we use the facts that \(\omega_0=2\pi/T\) and \(T=N\Delta t\).]

(g) We now use Euler’s relation: \(\exp(j2\pi ik) = \cos(2\pi ik)+js\sin(2\pi ik) = 1 + 0 = 1\) for all \(i\). Again, the sum over the time interval equals \(N\), and the expression in curly brackets equals 1.

(h) Thus, the expression in curly brackets equals 1 or 0 depending on whether \((m-n)\) is equal or not equal to \(iN\).
This conclusion is stated formally as:

\[
\left(\frac{1}{N}\right) \sum_{k=1}^{N} e^{j(m-n)\omega_0 k \Delta t} = \begin{cases} 
1, & \text{for } m-n=iN \\
0, & \text{for } m-n\neq iN
\end{cases}
\]

Essentially, the frequencies \( m-n=iN \) survive because their exponential terms equal 1. All other frequencies are eliminated because their exponential terms equal 0. Thus, the frequencies \( m-n=iN \) have a special meaning.
\[
Z(n) = \frac{1}{N} \sum_{k=1}^{N} \left[ \sum_{m=-\infty}^{\infty} Z(m) e^{j(m-n)\omega_0 k \Delta t} \right] = \sum_{m=-\infty}^{\infty} Z(m) \left\{ \left( \frac{1}{N} \right) \sum_{k=1}^{N} e^{j(m-n)\omega_0 k \Delta t} \right\}
\]

With respect to this equation:
1. most values of \( m \), i.e. those for which the sum of the exponential term equals 0, do not make any non-zero contribution to \( Z(n) \).
2. only the values of \( m \) for which \((m-n) = iN\), i.e. those for which the sum of the exponential term equals 1, make a non-zero contribution to \( Z(n) \).
Thus, we can use:

\[
\left( \frac{1}{N} \right) \sum_{k=1}^{N} e^{j(m-n)\omega_0 k \Delta t} = \begin{cases} 
1, & \text{for } m-n=iN \\
0, & \text{for } m-n\neq iN
\end{cases}
\]

to express:

\[
Z(n) = \frac{1}{N} \sum_{k=1}^{N} \left[ \sum_{m=-\infty}^{\infty} Z(m) e^{j(m-n)\omega_0 k \Delta t} \right] = \sum_{m=-\infty}^{\infty} Z(m) \left\{ \left( \frac{1}{N} \right) \sum_{k=1}^{N} e^{j(m-n)\omega_0 k \Delta t} \right\}
\]

as:

\[
Z(n) = \sum_{i=-\infty}^{\infty} Z(n + iN)
\]
This means that each term in the DFT of $X(k\Delta t)$ is the sum of a possibly infinite set of Fourier coefficients associated with the higher frequency components in $X(t)$. The higher frequency components are those corresponding to frequencies that are greater than $n$ by an amount $iN$. If $X(t)$ has no Fourier series components greater than $N/2$ (corresponding to the Nyquist frequency), then the apparent Fourier coefficients are correct. The DFT of $X(k\Delta t)$ yields correct results only if $X(t)$ is band-limited to frequencies below the Nyquist frequency. When $X(t)$ has higher frequencies, the high frequency components add to the low ones. This mixing is known as aliasing.
Once aliasing occurs, there is no way to separate out the true coefficient from the aliased one. This is why the lowpass cutoff frequency of the analog filter must be matched to the sampling rate such that the highest frequency with any appreciable power must be half of the sampling frequency, i.e. the Nyquist frequency. This is essential for analysis of sampled continuous data.
Example: Consider a cosine at the Nyquist frequency (F) sampled at the negative and positive peaks. If the frequency of the wave increases or decreases by a small increment a, sampling appears to be indistinguishable. In other words, it appears that a wave of frequency (F + a) will, after sampling, be confused with a wave of frequency (F – a).

![Wave Diagram]

Fig. 3.1. A cosine wave of frequency F (solid line) sampled at its Nyquist rate. A higher frequency (dotted) wave, frequency F + a, is shown sampled at the same rate. At the sample times it is indistinguishable from a lower frequency (dashed) wave, frequency F – a.
To determine all the frequencies that will be aliased onto F – a, start with

\[ Z(n) = \sum_{i=-\infty}^{\infty} Z(n + iN) \]

with \( n = (N/2) - k \), where \( k \) is an integer multiplier of \( \omega_0 \) corresponding to some arbitrary frequency component below the Nyquist frequency.

Note about this equation:

1. Index \( i \) extends from negative to positive infinity.

2. Since \( Z \) is complex, the representation of frequency \( n \) is both at \( Z(n) \) and \( Z(-n) \).
Let $n$ in this equation have the value of $[(N/2) - k]$:

$Z(n)$ becomes $Z[(N/2) - k]$

and

$Z(–n)$ becomes $Z[(-N/2)+k]$.

Then:

$$n + iN = (N/2) - k + iN = (i + 1/2)N - k$$

and

$$-n + iN = (-N/2) + k + iN = (i - 1/2)N + k$$
The two sequences are as follows:

\[
i \quad (i+1/2)N-k \quad (i-1/2)N+k
\]

- ** -- this is the term that shows aliasing
In real frequency terms:
A real term at \((N/2)+k\) comes from the complex pair at \((N/2)+k\) and \((-N/2)-k\).
A real term at \((3N/2)-k\) comes from the complex pair at \((3N/2)-k\) and \((-3N/2)+k\).
A real term at \((3N/2)+k\) comes from the complex pair at \((3N/2)+k\) and \((-3N/2)-k\).
A real term at \((5N/2)-k\) comes from the complex pair at \((5N/2)-k\) and \((-5N/2)+k\).
And so on.
In effect, the original Fourier representation of \(x(t)\) has been folded in accordion fashion about frequencies that are multiples of \(F\), and collapsed into the frequency region extending from 0 to \(F\) (which is also called the folding frequency).
In other words, the frequency axis is folded back onto the range 0-\(F\) in pieces of length \(F\).
Fig. 3.2. The occurrence of folding of the frequency (or $\pi$) axis due to sampling of a continuous signal. Frequency components of the original signal mixed with $\pi$'s on the $\pi$ axis are interpreted in the sampled version as belonging to the lowest frequency, an unmarked $\pi$.

Figure 1.6 The first 11 pleats of a Nyquist aliasing diagram.