X. Spectral Analysis
• Spectral analysis is the representation of time series data in terms of frequency content. It is based on the assumption that any arbitrary (non-periodic) time series may be decomposed into a sum of orthogonal periodic basis functions.

• Any function of frequency is called a spectrum.
• It is possible to represent non-periodic functions using any class of periodic functions.
• However, spectral analysis is nearly synonymous with Fourier analysis, which uses sines and cosines as the basis functions.
• Fourier analysis is a major tool in time series analysis since it has great utility in performing a number of operations.
The Fourier Transform

Fourier analysis begins with the Fourier transform of a time series. The Fourier transform can be expressed either in complex or real notation.
We begin with the complex form and derive the real form. The complex Fourier transform is defined as:

\[ Z(\omega) = \int_{-\infty}^{\infty} X(t)e^{-j\omega t} dt \]

where \( \omega \) represents frequency in radians/sec, and \( j \) is the square root of -1.
The Fourier transform of time function $X(t)$ can itself be transformed to give back the original function $X(t)$.

This is done by the **inverse Fourier transform**:

$$X(t) = \int_{-\infty}^{\infty} Z(\omega)e^{j\omega t} d\omega$$
$X(t)$ and $Z(\omega)$ are called a transform pair.

Either one can be obtained from the other.

In principle, every time function can be represented in the frequency domain as a spectral function, and can be recovered by way of the inverse Fourier transform.

In these definitions, both $t$ and $\omega$ range from negative to positive infinity. In practice, we must confine our analysis to a finite interval of time.
Fourier Series Representation

When we restrict our application of Fourier analysis to a finite time series of length $T$, only certain frequencies can be represented by Fourier analysis. Because of the orthogonality criterion, the only frequencies that can be represented are those for which length $T$ contains an integral number of cycles. Accordingly, the lowest frequency that can be represented is one in which one cycle fits exactly in $T$. 
The fundamental frequency of the time series is the lowest frequency that can be represented due to the finite observation length, and it is inversely dependent on $T$ as:

$$\omega_0 = \frac{2\pi}{T}$$

expressed in radians/sec.
Every other frequency $\omega$ in the series is an integer multiple of $\omega_0$, i.e. $\omega = n\omega_0$

When our time series is restricted in this manner to a finite length, then the Fourier transform of $X(t)$ becomes the Discrete Fourier Transform (DFT):

$$Z(n) = \frac{1}{T} \int_{0}^{T} X(t) e^{-j n \omega_0 t} dt, \quad -\infty < n < \infty$$

where $n$ is an integer multiplier of $\omega_0$. 
Note that the frequencies may be negative since \( n \) may be less than 0.

The inverse Fourier transform of \( Z(n) \) becomes the inverse DFT:

\[
X(t) = \sum_{n=-\infty}^{\infty} Z(n)e^{jn\omega_0 t}
\]

The right hand side of this equation is known as the complex Fourier series representation of \( X(t) \). The terms \( Z(n) \) are known as complex Fourier coefficients.
As indicated previously, the Fourier series may be expressed either in complex or real notation.

In order to convert to real notation, we introduce Euler's relation:

\[ e^{j\theta} = \cos \theta + j\sin \theta \]

The Fourier series representation of \( X(t) \) becomes:

\[
X(t) = \sum_{n=1}^{\infty} Z(n) [\cos(n\omega_0 t) + j\sin(n\omega_0 t)] \\
+ \sum_{n=1}^{\infty} Z(-n) [\cos(n\omega_0 t) - j\sin(n\omega_0 t)] \\
+ Z(0) [\cos(0) + j\sin(0)]
\]
By pairing negative and positive terms having the same value of n, we get:

\[
X(t) = Z(0) + \sum_{n=1}^{\infty} [Z(n)+Z(-n)] \cos(n\omega_0 t) + j[Z(n) - Z(-n)] \sin(n\omega_0 t)
\]

We see then that Fourier analysis allows \(X(t)\) to be represented by the sum of a set of sine and cosine waves of different amplitudes and harmonically related frequencies infinite in number.
We call the cosine terms the real Fourier coefficients and the sine terms the imaginary Fourier coefficients and define them as:

\[
A(n) = Z(n) + Z(-n)
\]

and

\[
B(n) = j[Z(n) - Z(-n)]
\]

Note that for \( n = 0 \), \( A(0) = 2 \times Z(0) \), and \( B(0) = 0 \).
It follows that:

\[ Z(n) = \frac{A(n) - jB(n)}{2} \]
\[ Z(-n) = \frac{A(n) + jB(n)}{2} \]

We see that \( Z(n) \) and \( Z(-n) \) are complex conjugates.

\( A(n) \) is the real part of \( Z(n) \) and \( B(n) \) is the imaginary part of \( Z(n) \).
Substituting gives the Fourier series representation in real notation:

\[ Z(n) = \frac{A(n) - jB(n)}{2} \]

\[ Z(-n) = \frac{A(n) + jB(n)}{2} \]

\[
X(t) = Z(0) + \sum_{n=1}^{\infty} \left[ Z(n) + Z(-n) \right] \cos(n\omega_0 t) + j[Z(n) - Z(-n)] \sin(n\omega_0 t)
\]

\[
X(t) = \frac{A(0)}{2} + \sum_{n=1}^{\infty} \left[ A(n) \cos(n\omega_0 t) + B(n) \sin(n\omega_0 t) \right]
\]
The real coefficients are given by:

\[ A(n) = \frac{1}{T} \int_0^T X(t) e^{-jn\omega_0 t} \, dt + \frac{1}{T} \int_0^T X(t) e^{jn\omega_0 t} \, dt \]

\[ = \frac{2}{T} \int_0^T X(t) \cos(n\omega_0 t) \, dt \]

And the imaginary coefficients are given by:

\[ B(n) = j \left[ \frac{1}{T} \int_0^T X(t) e^{-jn\omega_0 t} \, dt - \frac{1}{T} \int_0^T X(t) e^{jn\omega_0 t} \, dt \right] \]

\[ = \frac{2}{T} \int_0^T X(t) \sin(n\omega_0 t) \, dt \]
If we are given any complex (aperiodic) time series, we may obtain the Fourier series representation by

\[ X(t) = \frac{A(0)}{2} + \sum_{n=1}^{\infty} [A(n) \cos(n\omega_0 t) + B(n) \sin(n\omega_0 t)] \]

where we determine the real and imaginary coefficients by

\[ A(n) = \frac{1}{T} \int_{0}^{T} X(t)e^{-jn\omega_0 t} dt + \frac{1}{T} \int_{0}^{T} X(t)e^{jn\omega_0 t} dt \]
\[ = \frac{2}{T} \int_{0}^{T} X(t) \cos(n\omega_0 t) dt \]

\[ B(n) = j[\frac{1}{T} \int_{0}^{T} X(t)e^{-jn\omega_0 t} dt - \frac{1}{T} \int_{0}^{T} X(t)e^{jn\omega_0 t} dt] \]
\[ = \frac{2}{T} \int_{0}^{T} X(t) \sin(n\omega_0 t) dt \]
Examples of Fourier Series Representations

Remember the definition of the fundamental frequency:

\[ \omega_0 = \frac{2\pi}{T} \]

For any frequency \( n\omega_0 \), the Fourier basis functions are \( \cos(n*2\pi t/T) \) and \( \sin(n*2\pi t/T) \).

For example, at the fundamental frequency, the Fourier basis functions are \( \cos(1*2\pi t/T) \) and \( \sin(1*2\pi t/T) \).
Example 1:

a. Let the time series $X(t)$ be a cosine wave with amplitude of 1 (range=+/- 0.5)

b. Let the time series $X(t)$ be a sine wave with amplitude of 1 (range=+/- 0.5.)
$1 \cos(1 \cdot 2\pi \cdot t/T)$

$1 \sin(1 \cdot 2\pi \cdot t/T)$
Example 2:

a. Let $X(t) = [\cos(1 \cdot 2\pi t/T) + \sin(1 \cdot 2\pi t/T)]$

b. Let $X(t) = [5\cos(1 \cdot 2\pi t/T) + 3\sin(1 \cdot 2\pi t/T)]$
\[ x(t) = 1 \cos(1 \times 2 \pi t/T) + 1 \sin(1 \times 2 \pi t/T) \]

\[ x(t) = 5 \cos(1 \times 2 \pi t/T) + 3 \sin(1 \times 2 \pi t/T) \]
Example 4:

Let $X(t)$ be the sum of cosine and sine waves at 3 different frequencies:

$$X(t) = 5\cos(1 \times 2\pi/T) + 3\sin(1 \times 2\pi/T)$$
$$+ 4\cos(3 \times 2\pi/T) + 7\sin(3 \times 2\pi/T)$$
$$+ 3\cos(7 \times 2\pi/T) + 2\sin(7 \times 2\pi/T)$$

The different components are shown in the next slide.
The Fourier series representation shows components at the appropriate frequencies with the appropriate amplitudes.
What happens to components at the non-integer multiples of $\omega_0$?

Consider an infinite length time series that is periodic. It only has one frequency component. When we obtain a finite length sample, then we may choose $T$ to correspond to one period of the periodic time series. In that case, the spectrum of $T$ is the same as that of the infinite time series.

Consider an infinite length time series that is aperiodic. It has multiple frequency components, possibly an infinite number of them. When we obtain a finite length sample, we still have a finite interval $T$. Although the time series may contain components at any frequency, we can only resolve those components with frequencies that are integer multiples of $\omega_0$, which is determined by $T$.

What happens to the other frequency components?

The non-integer multiples of $\omega_0$ get misrepresented as one of the integer multiples. All frequency components in a range between $(n - 1/2)\omega_0$ and $(n + 1/2)\omega_0$, i.e. a band $\omega_0$ wide centered at $n\omega_0$, are lumped into a single frequency $n\omega_0$. 