where \( r = -n, \ldots, 0, \ldots, n-1 \) and \( N = 2n \). The highest frequency is \( 1/(2\Delta t) \) c/sec which corresponds to a period of two sampling intervals, i.e., \( 2\Delta t \), and is called the Nyquist frequency \( f_N \) (note that this "N" represents Nyquist, not \( T/\Delta t \)).

The calculations for the finite Fourier transform consist of forming the average products of the discrete sample series \( X_r \) with discrete cosine and sine series of different frequencies:

\[
A_k = \frac{1}{N} \sum_{r=-n}^{n-1} X_r \cos \frac{2\pi kr}{N}
\]

\[
B_k = \frac{1}{N} \sum_{r=-n}^{n-1} X_r \sin \frac{2\pi kr}{N}
\]

for \( k = 1, \ldots, n \) for \( A_k \) and \( k = 1, \ldots, (n-1) \) for \( B_k \). At zero frequency (\( k = 0 \)), this becomes

\[
A_0 = \frac{1}{N} \sum_{r=-n}^{n-1} X_r
\]

representing the mean or average value of the \( X_r \), also called the D.C. component of the signal. The mean square value or average intensity of the signal \( X_r \) is defined by

\[
\frac{1}{N} \sum_{r=-n}^{n-1} X_r^2
\]

After defining \( S_k^2 = A_k^2 + B_k^2 \) (and \( S_0^2 = A_0^2 \)) we get the relation

\[
\frac{1}{N} \sum_{r=-n}^{n-1} X_r^2 = S_0^2 + 2 \sum_{k=1}^{n-1} S_k^2 + S_0^2
\]

which states that the mean square value or the average intensity of a signal can be decomposed into contributions from each harmonic (Parseval's theorem). By subtracting the mean \( S_0 (= A_0 \) B.7) from the data we get the variance or average intensity (or average A.C. power).

\[
\sigma^2 = \frac{1}{N} \sum_{r=-n}^{n-1} (X_r - S_0)^2 = 2 \sum_{k=1}^{n-1} S_k^2 + S_0^2
\]

This decomposition of the total variance gives the average intensity or variance at each harmonic. This display of the variance components as a function of frequency is called the periodogram (Fig. 2).

### C. Random Processes

Like many other biophysical phenomena the electroencephalogram shows a more or less irregular pattern and, therefore, can be described only by average properties. It is quite obvious that EEG records belong to the category of random data which cannot be defined by explicit mathematical relations (see Subsection B). Random data are described in statistical terms, i.e., by probability distributions and averages.
such as means, variances, covariances and spectra, higher order moments, etc.

The analytical approach is to define a \textit{random process \{X(t)\}} which generates
"realizations", which may be thought of as random functions, having statistical
features as close as possible to the observed data. A random (or stochastic) process
represents the infinite collection of all possible sample functions which the random
phenomenon might have produced during infinite time. This collection is called an
\textit{ensemble}. Hence a sample record of EEG data may be regarded as one realization
of a random process. It must be emphasized that a random process is a mathematical
model rather than a physical reality. Using this model we may describe certain
average properties of the data by mathematical terms based on a firm statistical
theory.

Any random variable can be described primarily by its mean value and its distribution
around the mean. In the case that distribution is \textit{normal} or \textit{Gaussian}, the mean
and the variance are sufficient descriptors of the random variable as long as its ob-
served values are mutually independent. However, in the case of a time series X(t),
its values are not likely to be independent from each other and thus we have to also
take into account that interdependence; it is expressed by the autocovariance of
X(t) (see Subsection E).

A most important class of random processes has the property that the auto-
covariances of single amplitude functions depend only on time differences rather
than on time itself. Such processes are called \textit{(weakly) stationary}. A stationary random
process may be either Gaussian or non-Gaussian. A stationary Gaussian random
process is completely described by its mean, variance (Subsection D) and auto-
covariance (Subsection E) or the Fourier transform of the autocovariance, which
is the power spectrum (Subsection F). For non-Gaussian stationary random processes
an extension of analysis to higher order moments is possible (Subsections D and H).
Non-stationary random processes present considerable analytical difficulties.
Therefore the assumption of stationarity is most important. Unfortunately the
bioelectrical activity of the brain frequently changes its average behaviour with time
and the assumption of stationarity is often more or less violated. In these cases the
use of a stationarity model brings only more or less good approximations. Such
approximations, however, may be of use all the same (Granger and Hatanaka 1964),
as will be shown by some examples (Subsection G.4).

From the point of view of the underlying statistical theory, \textit{ergodicity} is a further
important condition for the application of the model of a stationary random process.
Such a process is ergodic if time averages are equal to the corresponding ensemble
averages, \textit{i.e.}, if we get from one infinite sample function all the information about
the ensemble. The ergodic condition therefore means that one sample function is
representative for the entire collection of possible realizations of the random process.
For a discussion of this property we refer to the textbooks on time series analysis
(Bendat and Piersol 1966; Jenkins and Watts 1968; Taubenheim 1969; Hannan
1970, etc.).
NUMERICAL SPECTRAL ANALYSIS

D. AMPLITUDE DISTRIBUTION AND CENTRAL MOMENTS

The relative frequencies of occurrence of equal instantaneous amplitude values of a data record during the observation epoch T may be sorted into a bar chart called amplitude histogram. If this process is ergodic it provides an estimate of the probability density function and already contains valuable information about the characteristics of the EEG sample record. It not only gives an estimate of the mean

\[ m_1 = \overline{X} = E\{x\} \]  \hspace{1cm} (D.1)

(where \( E\{\} \) represents the expected value of the quantity within the brackets when measured over an infinite period of time), and of the variance (= second central moment \( m_2 \)) or average intensity

\[ m_2 = \sigma^2 = E\{(X - \overline{X})^2\} \]  \hspace{1cm} (D.2)

but also of the higher central moments. From the third central moment

\[ m_3 = E\{(X - \overline{X})^3\} \]  \hspace{1cm} (D.3)

we derive the skewness factor

\[ \beta_1 = \frac{m_3}{(m_2)^{\frac{3}{2}}} \]  \hspace{1cm} (D.4)

which in the case of a symmetrical distribution is zero. From the fourth central moment

\[ m_4 = E\{(X - \overline{X})^4\} \]  \hspace{1cm} (D.5)

one gets to kurtosis

\[ \beta_2 = \frac{m_4}{(m_2)^2} \]  \hspace{1cm} (D.6)

which provides a measure of the relative peakedness or flatness of a distribution. For a Gaussian or normal distribution

\[ p(X) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{X - \overline{X}}{\sigma}\right)^2\right] \]  \hspace{1cm} (D.7)

where \( \exp[a] = e^a \), \( e = 2.71828 \ldots \); the odd higher moments are zero and the even ones have known values, e.g., \( \beta_2 = 3 \). The quantity \( \beta_2 - 3 \), called excess of kurtosis, is sometimes preferred. Assumption of a normal distribution is important for deriving theoretical statistical measures for various analytical methods in time series analysis, especially for spectral analysis. The higher order moments characterize deviations from a normal or Gaussian distribution and therefore provide some rough tests of validity for using mathematical models which assume Gaussian data.

Variance, skewness and kurtosis may be also of direct interest. Some EEG patterns, such as the \( \mu \)-rhythm, fourteen-and-six per second positive spikes, series of sharp waves, etc. contribute to a skewed (i.e., asymmetrical) amplitude distribution, whereas symmetrical paroxysmal events (as well as, unfortunately, some common artefacts) produce significant kurtosis.
Amplitude histograms of EEG data were investigated by Lion and Winter (1953), Kozhevnikov (1958a), Saunders (1963), Elul (1967, 1968, 1969a) and by Sato et al. (1970). Some recent studies showed that in contrast to the general opinion, even the spontaneous waking alpha-EEG deviates more or less from a normal amplitude distribution in a substantial number of subjects (Dumermuth et al. 1972b). Skewness and kurtosis were at least ten times larger than the values measured from a random noise generator (Fig. 3). In adult sleep, skewness and kurtosis characterize certain sleep stages and therefore may contribute to solving the problem of quantifying sleep activity (Fig. 4) (Dumermuth et al. 1972a).

Significant values of skewness and especially of kurtosis may also indicate non-stationarity (Fig. 5).

E. CORRELATION ANALYSIS

Statistical properties of one channel of EEG or relations between two or more channels recorded simultaneously from different brain areas may be investigated by correlation analysis. It was Norbert Wiener who suggested the application of this technique to the electroencephalogram. The first investigations were then carried out at M.I.T. (Brazier et al. 1952, 1956; Barlow 1959; Brazier 1963). For further information see Brazier 1961, Rosenblith 1962, Livanov and Rusinov 1965. Section V of this Volume is devoted to correlation analysis; only some of its principles necessary to understand the relations between correlation analysis and spectral analysis will be sketched here.

The autocovariance of a stationary random signal with zero mean $\langle X = 0 \rangle$ (cf. Eqn. D.1)