Dynamical Systems in One and Two Dimensions: A Geometrical Approach

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Abstract. This chapter is intended as an introduction or tutorial to nonlinear dynamical systems in one and two dimensions with an emphasis on keeping the mathematics as elementary as possible. By its nature such an approach does not have the mathematical rigor that can be found in most textbooks dealing with this topic. On the other hand it may allow readers with a less extensive background in math to develop an intuitive understanding of the rich variety of phenomena that can be described and modeled by nonlinear dynamical systems. Even though this chapter does not deal explicitly with applications – except for the modeling of human limb movements with nonlinear oscillators in the last section – it nevertheless provides the basic concepts and modeling strategies all applications are build upon. The chapter is divided into two major parts that deal with one- and two-dimensional systems, respectively. Main emphasis is put on the dynamical features that can be obtained from graphs in phase space and plots of the potential landscape, rather than equations and their solutions. After discussing linear systems in both sections, we apply the knowledge gained to their nonlinear counterparts and introduce the concepts of stability and multistability, bifurcation types and hysteresis, hetero- and homoclinic orbits as well as limit cycles, and elaborate on the role of nonlinear terms in oscillators.

1 One-Dimensional Dynamical Systems

The one-dimensional dynamical systems we are dealing with here are systems that can be written in the form

$$\frac{dx(t)}{dt} = \dot{x}(t) = f[x(t), \{\lambda\}] \tag{1}$$

In (1) $x(t)$ is a *function*, which, as indicated by its argument, depends on the *variable* t representing time. The left and middle part of (1) are two ways of expressing

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how the function $x(t)$ changes when its variable t is varied, in mathematical terms called the *derivative* of $x(t)$ with respect to t. The notation in the middle part, with a dot on top of the variable, $\dot{x}(t)$, is used in physics as a short form of a derivative with respect to time. The right-hand side of (1), $f[x(t),\{\lambda\}]$, can be any function of $x(t)$ but we will restrict ourselves to cases where f is a low-order polynomial or trigonometric function of $x(t)$. Finally, $\{\lambda\}$ represents a set of *parameters* that allow for controlling the system's dynamical properties. So far we have explicitly spelled out the function with its argument, from now on we shall drop the latter in order to simplify the notation. However, we always have to keep in mind that $x = x(t)$ is not simply a variable but a function of time.

In common terminology (1) is an ordinary autonomous differential equation of first order. It is a *differential equation* because it represents a relation between a function (here x) and its derivatives (here $\dot{x}$). It is called *ordinary* because it contains derivatives only with respect to one variable (here t) in contrast to *partial* differential equations that have derivatives to more than one variable – spatial coordinates in addition to time, for instance – which are much more difficult to deal with and not of our concern here. Equation (1) is *autonomous* because on its right-hand side the variable t does not appear explicitly. Systems that have an explicit dependence on time are called *non-autonomous* or *driven*. Finally, the equation is of *first order* because it only contains a first derivative with respect to t; we shall discuss second order systems in sect. 2.

It should be pointed out that (1) is by no means the most general one-dimensional dynamical system one can think of. As already mentioned, it does not explicitly depend on time, which can also be interpreted as decoupled from any environment, hence autonomous. Equally important, the change $\dot{x}$ at a given time t only depends on the state of the system at the same time $x(t)$, not at a state in its past $x(t-\tau)$ or its future $x(t+\tau)$. Whereas the latter is quite peculiar because such systems would violate causality, one of the most basic principles in physics, the former simply means that system has a memory of its past. We shall not deal with such systems here; in all our cases the change in a system will only depend on its current state, a property called *markovian*.

A function $x(t)$ which satisfies (1) is called a *solution* of the differential equation. As we shall see below there is never a single solution but always infinitely many and all of them together built up the *general solution*. For most nonlinear differential equations it is not possible to write down the general solution in a closed analytical form, which is the bad news. The good news, however, is that there are easy ways to figure out the dynamical properties and to obtain a good understanding of the possible solutions without doing sophisticated math or solving any equations.

1.1 Linear Systems

The only linear one-dimensional system that is relevant is the equation of continuous growth

$$\dot{x} = \lambda x \tag{2}$$

where the change in the system $\dot{x}$ is proportional to state x. For example, the more members of a given species exist, the more offsprings they produce and the faster the population grows given an environment with unlimited resources. If we want to know the time dependence of this growth explicitly, we have to find the solutions of (2), which can be done mathematically but in this case it is even easier to make an educated guess and then verify its correctness. To solve (2) we have to find a function $x(t)$ that is essentially the same as its derivative $\dot{x}$ times a constant λ. The family of functions with this property are the exponentials and if we try

$$x(t) = e^{\lambda t} \qquad \text{we find} \qquad \dot{x}(t) = \lambda\, e^{\lambda t} \qquad \text{hence} \qquad \dot{x} = \lambda\, x \tag{3}$$

and therefore $x(t)$ is a solution. In fact if we multiply the exponential by a constant c it also satisfies (2)

$$x(t) = c\, e^{\lambda t} \qquad \text{we find} \qquad \dot{x}(t) = c\,\lambda\, e^{\lambda t} \qquad \text{and still} \qquad \dot{x} = \lambda\, x \tag{4}$$

But now these are infinitely many functions – we have found the general solution of (2) – and we leave it to the mathematicians to prove that these are the only functions that fulfill (2) and that we have found all of them, i.e. *uniqueness* and *completeness* of the solutions. It turns out that the general solution of a dynamical system of n^{th} order has n open constants and as we are dealing with one-dimension systems here we have one open constant: the c in the above solution. The constant c can be determined if we know the state of the system at a given time t, for instance $x(t=0) = x_0$

$$x(t=0) = x_0 = c\, e^0 \quad \rightarrow \quad c = x_0 \tag{5}$$

where x_0 is called the *initial condition*. Figure 1 shows plots of the solutions of (2) for different initial conditions and parameter values $\lambda < 0$, $\lambda = 0$ and $\lambda > 0$.

We now turn to the question whether it is possible to get an idea of the dynamical properties of (2) or (1) *without* calculating solutions, which, as mentioned above, is not possible in general anyway. We start with (2) as we know the solution in this

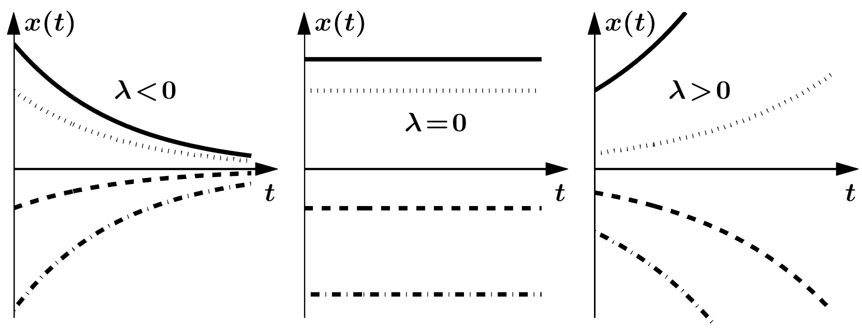

Fig. 1 Solutions $x(t)$ for the equation of continuous growth (2) for different initial conditions x_0 (solid, dashed, dotted and dash-dotted) and parameter values $\lambda < 0$, $\lambda = 0$ and $\lambda > 0$ on the left, in the middle and on the right, respectively.

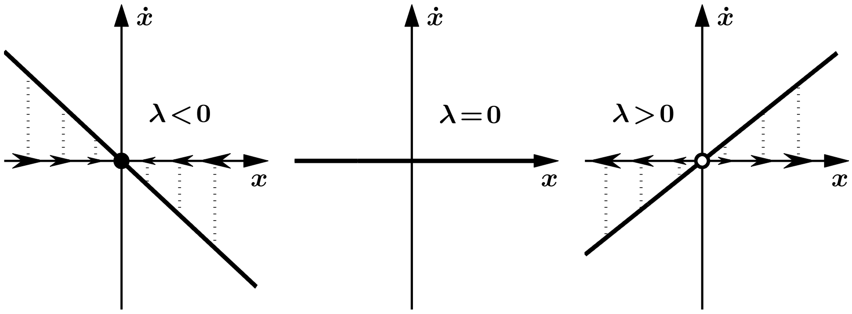

Fig. 2 Phase space plots, $\dot{x}$ as a function of x, for the equation of continuous growth (2) for the cases $\lambda < 0$, $\lambda = 0$ and $\lambda > 0$ on the left, in the middle and on the right, respectively.

case and now plot $\dot{x}$ as a function of x, a representation called a *phase space plot* and shown in fig. 2, again for $\lambda < 0$, $\lambda = 0$ and $\lambda > 0$. The graphs are straight lines given by $\dot{x} = \lambda\, x$ with a negative, vanishing and positive slope, respectively. So what can we learn from these graphs? The easiest is the one in the middle corresponding to $\dot{x} = 0$, which means there are no changes in the system. Where ever we start initially we stay there, a quite boring case.

Next we turn to the plot on the left, $\lambda < 0$, for which the phase space plot is a straight line with a negative slope. So for any state $x < 0$ the change $\dot{x}$ is positive, the system evolves to the right. Moreover, the more negative the state x the bigger the change $\dot{x}$ towards the origin as indicated by the direction and size of the arrows on the horizontal axis. In contrast, for any initial positive state $x > 0$ the change $\dot{x}$ is negative and the system evolves towards the left. In both cases it is approaching the origin and the closer it gets the more it slows down. For the system (2) with $\lambda < 0$ all *trajectories* evolve towards the origin, which is therefore called a *stable fixed point* or *attractor*. Fixed points and their stability are most important properties of dynamical systems, in particular for nonlinear systems as we shall see later. In phase space plots like fig. 2 stable fixed points are indicated by solid circles.

On the right in fig. 2 the case for $\lambda > 0$ is depicted. Here, for any positive (negative) state x the change $\dot{x}$ is also positive (negative) as indicated by the arrows and the system moves away from the origin in both direction. Therefore, the origin in this case is an *unstable fixed point* or *repeller* and indicated by an open circle in the phase space plot. Finally, coming back to $\lambda = 0$ shown in the middle of fig. 2, all points on the horizontal axis are fixed points. However, they are neither attracting nor repelling and are therefore called *neutrally stable*.

1.2 Nonlinear Systems: First Steps

The concepts discussed in the previous section for the linear equation of continuous growth can immediately be applied to nonlinear systems in one dimension. To be most explicit we treat an example known as the *logistic equation*

$$\dot{x} = \lambda x - x^2 \tag{6}$$

The graph of this function is a parabola which opens downwards, it has one intersection with the horizontal axis at the origin and another one at $x = \lambda$ as shown in fig. 3.

These intersections between the graph and the horizontal axis are most important because they are the fixed points of the system, i.e. the values of x for which $\dot{x} = 0$ is fulfilled. For the case $\lambda < 0$, shown on the left in fig. 3, the graph intersects the negative x-axis with a positive slope. As we have seen above – and of course one can apply the reasoning regarding the state and its change here again – such a slope means that the system is moving away from this point, which is therefore classified as an unstable fixed point or repeller. The opposite is case for the fixed point at the origin. The flow moves towards this location from both side, so it is stable or an attractor. Corresponding arguments can be made for $\lambda > 0$ shown on the right in fig. 3.

An interesting case is $\lambda = 0$ shown in the middle of fig. 3. Here the slope vanishes, a case we previously called neutrally stable. However, by inspecting the state and change in the vicinity of the origin, it is easily determined that the flow moves towards this location if we are on the positive x-axis and away from it when x is negative. Such points are called *half-stable* or *saddle points* and denoted by half-filled circles.

As a second example we discuss the cubic equation

$$\dot{x} = \lambda x - x^3 \tag{7}$$

From the graph of this function, shown in fig. 4, it is evident that for $\lambda \leq 0$ there is one stable fixed point at the origin which becomes unstable when λ is increased to positive values and at the same time two stable fixed points appear to its right and left. Such a situation, where more than one stable state exist in a system is called *multistability*, in the present case of two stable fixed points *bistability*, an inherently nonlinear property which does not exist in linear systems. Moreover, (7)

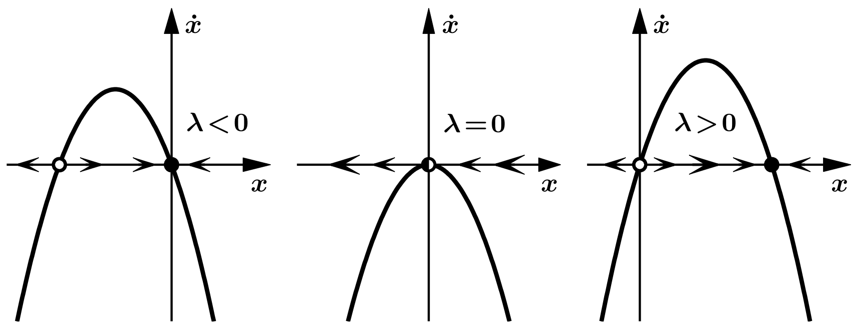

Fig. 3 Phase space plots, $\dot{x}$ as a function of x, for the logistic equation (6) for the cases $\lambda < 0$, $\lambda = 0$ and $\lambda > 0$ on the left, in the middle and on the right, respectively.

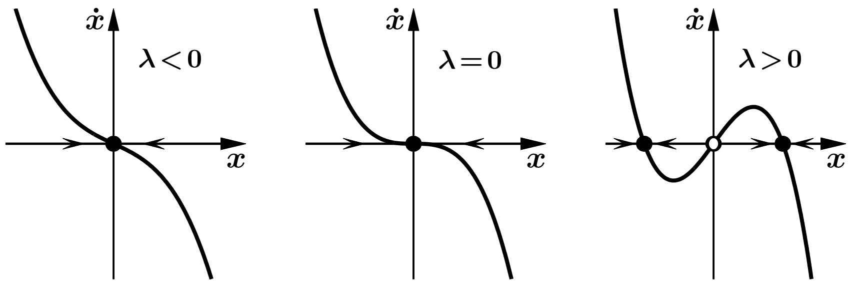

Fig. 4 Phase space plots, $\dot{x}$ as a function of x, for the cubic equation (7) for the cases $\lambda < 0$, $\lambda = 0$ and $\lambda > 0$ on the left, in the middle and on the right, respectively.

becomes bistable when the parameter λ switches from negative to positive values. When this happens, the change in the system's dynamical behavior is not gradual but *qualitative*. A system, which was formerly *monostable* with a single attractor at the origin, now has become *bistable* with three fixed points, two of them stable and the origin having switched from an attractor to a repeller. It is this kind of qualitative change in behavior when a parameter exceeds a certain threshold that makes nonlinear differential equations the favorite modeling tool to describe the transition phenomena we observe in nature.

1.3 Potential Functions

So far we derived the dynamical properties of linear and nonlinear systems from their phase space plots. There is another, arguably even more intuitive way to find out about a system's behavior, which is by means of potential functions. In one-dimensional systems the potential is defined by

$$\dot{x} = f(x) = -\frac{dV}{dx} \quad \rightarrow \quad V(x) = -\int f(x)\,dx + c \tag{8}$$

In words: the negative derivative of the potential function is the right-hand side of the differential equation. All one-dimensional systems have a potential, even though it may not be possible to write it down in a closed analytical form. For higher dimensional systems the existence of a potential is more the exception than the rule as we shall see in sect. 2.5.

From its definition (8) it is obvious that the change in state $\dot{x}$ is equal to the negative slope of the potential function. First, this implies that the system always moves in the direction where the potential is decreasing and second, that the fixed points of the system are located at the extrema of the potential, where minima correspond to stable and maxima to unstable fixed points. The dynamics of a system can be seen as the overdamped motion of a particle the landscape of the potential. One can think of an overdamped motion as the movement of a particle in a thick or viscous fluid like honey. If it reaches a minimum it will stick there, it will not oscillate back and forth.

Examples

1. $\dot{x} = \lambda x = -\dfrac{dV}{dx} \quad \rightarrow \quad V(x) = -\int \lambda x dx + c = -\dfrac{1}{2}\lambda x^2 \underbrace{+c}_{=0}$

   The familiar linear equation. Plots of $\dot{x}$ and the corresponding potential V as functions of x are shown in fig. 5 for the cases $\lambda < 0$ (left) and $\lambda > 0$ (middle);

2. $\dot{x} = x - x^2 = -\dfrac{dV}{dx} \quad \rightarrow \quad V(x) = -\dfrac{1}{2}x^2 + \dfrac{1}{3}x^3$

   A special case of the logistic equation. The potential in this case is a cubic function shown in fig. 5 (right);

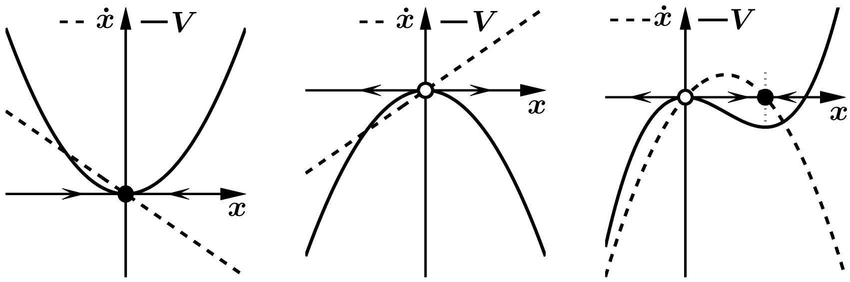

Fig. 5 Graphs of $\dot{x}$ (dashed) and $V(x)$ (solid) for the linear equation ($\lambda < 0$ left, $\lambda > 0$ middle) and for the logistic equation (right).

3. $\dot{x} = \lambda x - x^3 \quad \rightarrow \quad V(x) = -\frac{1}{2}\lambda x^2 + \frac{1}{4}x^4$

   The cubic equation for which graphs and potential functions are shown in fig. 6. Depending on the sign of the parameter λ this system has either a single attractor at the origin or a pair of stable fixed points and one repeller.

4. $\dot{x} = \lambda + x - x^3 \quad \rightarrow \quad V(x) = -\lambda x - \frac{1}{2}x^2 + \frac{1}{4}x^4$

   For the case $\lambda = 0$ this equation is a special case of the cubic equation we have dealt with above, namely $\dot{x} = x - x^3$. The phase space plots for this special case are shown in fig. 7 in the left column. The top row in this figure shows what is happening when we increase λ from zero to positive values. We are simply adding a constant, so the graph gets shifted upwards. Correspondingly, when we decrease λ from zero to negative values the graph gets shifted downwards, as shown in the bottom row in fig. 7.

   The important point in this context is the number of intersections of the graphs with the horizontal axis, i.e. the number of fixed points. The special case with $\lambda = 0$ has three as we know and if we increase or decrease λ only slightly this number stays the same. However, there are certain values of λ, for which one of

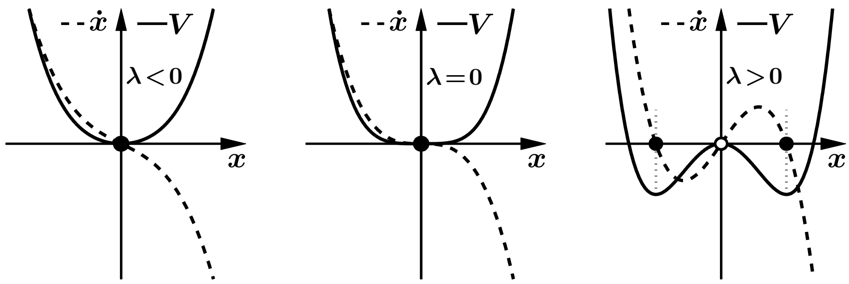

Fig. 6 Graph of $\dot{x}$ (dashed) and $V(x)$ (solid) for the cubic equation for different values of λ.

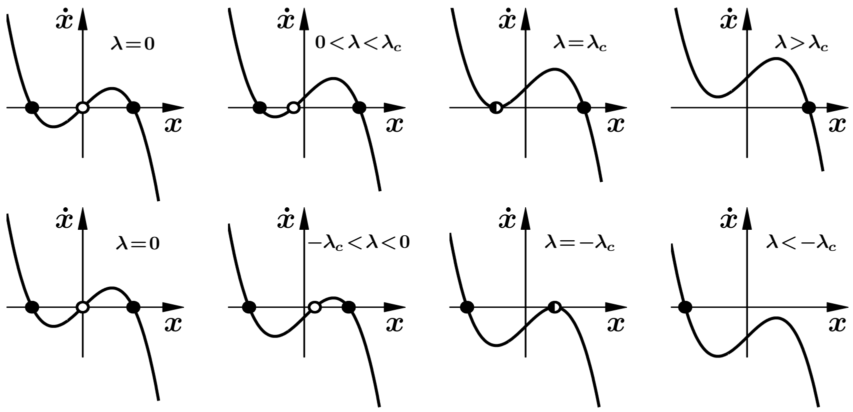

Fig. 7 Phase space plots for $\dot{x} = \lambda + x - x^3$. For positive (negative) values of λ the graphs are shifted up (down) with respect to the point symmetric case $\lambda = 0$ (left column). The fixed point skeleton changes at the critical parameter values $\pm\lambda_c$.

the extrema is located on the horizontal axis and the system has only two fixed points as can be seen in the third column in fig. 7. We call these the critical values for the parameter, $\pm\lambda_c$. A further increase or decrease beyond these critical values leaves the system with only one fixed point as shown in the rightmost column. Obviously, a qualitative change in the system occurs at the parameter values $\pm\lambda_c$ when a transition from three fixed points to one fixed point takes place.

A plot of the potential functions where the parameter is varied from $\lambda < -\lambda_c$ to $\lambda > \lambda_c$ is shown in fig. 8. In the graph on the top left for $\lambda < -\lambda_c$ the potential has a single minimum corresponding to a stable fixed point, as indicated by the gray ball, and the trajectories from all initial conditions end there. If λ is increased a half-stable fixed point emerges at $\lambda = -\lambda_c$ and splits into a stable and unstable fixed point, i.e. a local minimum and maximum when the parameter exceeds this threshold. However, there is still the local minimum for negative values of x and the system, represented by the gray ball, will remain there. It takes an increase in λ beyond λ_c in the bottom row before this minimum disappears and the system switches to the only remaining fixed point on the right. Most importantly, the dynamical behavior is different if we start with a $\lambda > \lambda_c$, as in the graph at the bottom right and decrease the control parameter. Now the gray ball will stay at positive values of x until the critical value $-\lambda_c$ is passed and the system switches to the left. The state of the system does not only depend on the value of the control parameter but also on its history of parameter changes – it has a form of memory. This important and wide spread phenomenon is called *hysteresis* and we shall come back to it in sect. 1.4.

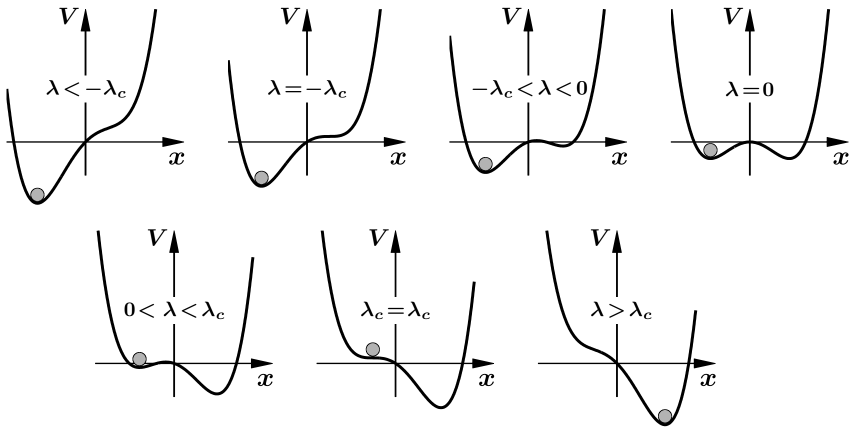

Fig. 8 Potential functions for $\dot{x} = \lambda + x - x^3$ for parameter values $\lambda < -\lambda_c$ (top left) to $\lambda > \lambda_c$ (bottom right). If a system, indicated by the gray ball, is initially in the left minimum, λ has to increase beyond λ_c before a switch to the right minimum takes place. In contrast, if the system is initially in the right minimum, λ has to decrease beyond $-\lambda_c$ before a switch occurs. The system shows hysteresis.

1.4 Bifurcation Types

One major difference between linear and nonlinear systems is that the latter can undergo qualitative changes when a parameter exceeds a critical value. So far we have characterized the properties of dynamical systems by phase space plots and potential functions for different values of the control parameter, but it is also possible to display the locations and stability of fixed points as a function of the parameter in a single plot, called a *bifurcation diagram*. In these diagrams the locations of stable fixed points are represented by solid lines, unstable fixed points are shown dashed. We shall also use solid, open and half-filled circles to mark stable, unstable and half-stable fixed points, respectively.

There is a quite limited number of ways how such qualitative changes, also called *bifurcations*, can take place in one-dimensional systems. In fact, there are four basic types of bifurcations known as *saddle-node, transcritical*, and *super-* and *subcritical* *pitchfork bifurcation*, which we shall discuss. For each type we are going to show a plot with the graphs in phase space at the top, the potentials in the bottom row, and in-between the bifurcation diagram with the fixed point locations $\tilde{x}$ as functions of the control parameter λ.

Saddle-Node Bifurcation

The prototype of a system that undergoes a saddle-node bifurcation is given by

$$\dot{x} = \lambda + x^2 \quad \rightarrow \quad \tilde{x}_{1,2} = \pm\sqrt{-\lambda} \tag{9}$$

The graph in phase space for (9) is a parabola that open upwards. For negative values of λ one stable and one unstable fixed point exist, which collide and annihilate when λ is increased above zero. There are no fixed points in this system for positive values of λ. Phase space plots, potentials and a bifurcation diagram for (9) are shown in fig. 9.

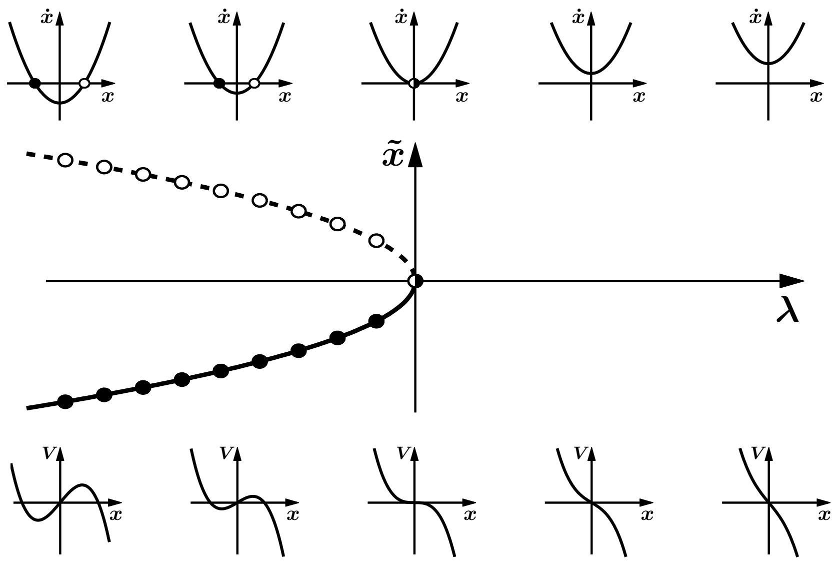

Fig. 9 Saddle-node bifurcation: a stable and unstable fixed point collide and annihilate. Top: phase space plots; middle: bifurcation diagram; bottom: potential functions.

Transcritical Bifurcation

The transcritical bifurcation is given by

$$\dot{x} = \lambda x + x^2 \quad \rightarrow \quad \tilde{x}_1 = 0, \;\; \tilde{x}_2 = \lambda \tag{10}$$

and summarized in fig. 10. For all parameter values, except the bifurcation point $\lambda = 0$, the system has a stable and an unstable fixed point. The bifurcation diagram

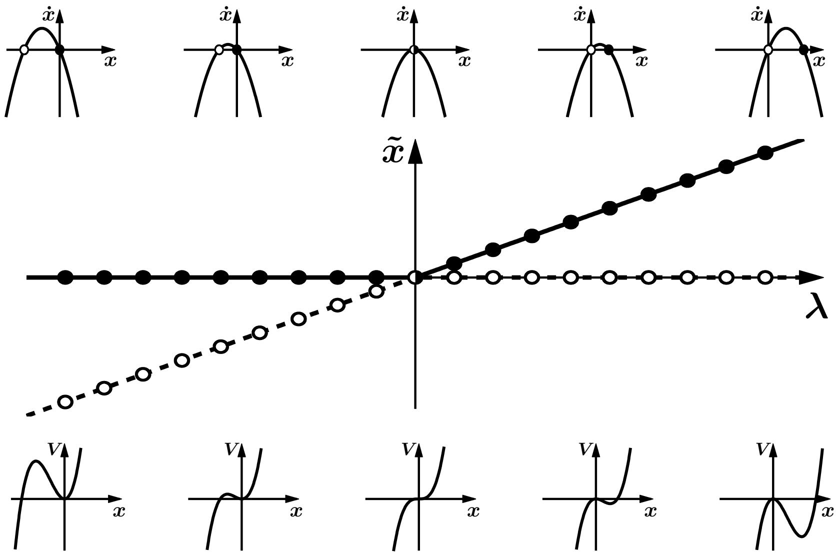

Fig. 10 Transcritical bifurcation: a stable and an unstable fixed point exchange stability. Top: phase space plots; middle: bifurcation diagram; bottom: potential functions.

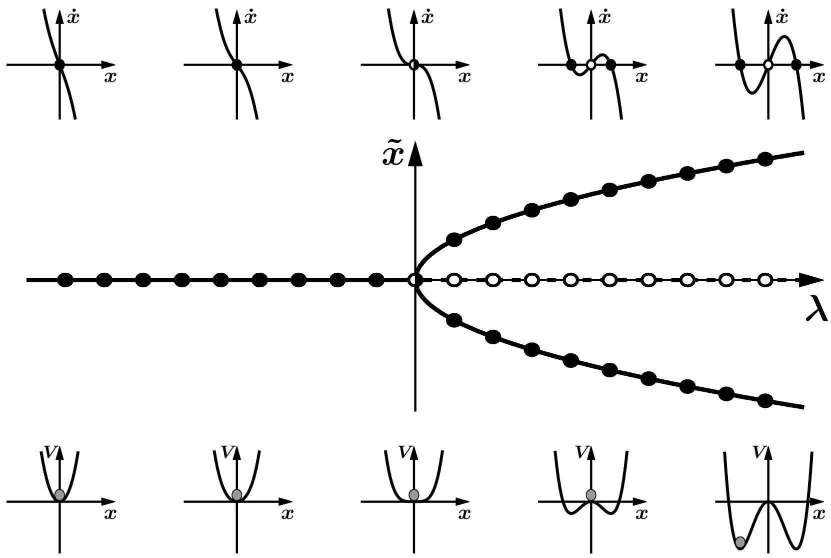

Fig. 11 Supercritical pitchfork bifurcation: a stable fixed point becomes unstable and two new stable fixed points arise. Top: phase space plots; middle: bifurcation diagram; bottom: potential functions.

consists of two straight lines, one at $\tilde{x} = 0$ and one with a slope of one. When these lines intersect at the origin they exchange stability, i.e. former stable fixed points along the horizontal line become unstable and the repellers along the line with slope one become attractors.

Supercritical Pitchfork Bifurcation

The supercritical pitchfork bifurcation is visualized in fig. 11 and is prototypically given by

$$\dot{x} = \lambda x - x^3 \quad \rightarrow \quad \tilde{x}_1 = 0,\ \tilde{x}_{2,3} = \pm\sqrt{\lambda} \tag{11}$$

The supercritical pitchfork bifurcation is the main mechanism for switches between mono- and bistability in nonlinear systems. A single stable fixed point at the origin becomes unstable and a pair of stable fixed points appears symmetrically around $\tilde{x} = 0$. In terms of symmetry this system has an interesting property: the differential equation (11) is invariant if we substitute x by $-x$. This can also be seen in the phase space plots, which all have a point symmetry with respect to the origin, and in the plots of the potential, which have a mirror symmetry with respect to the vertical axis. If we prepare the system with a parameter $\lambda < 0$ it will settle down at the only fixed point, the minimum of the potential at $x = 0$, as indicated by the gray ball in fig. 11 (bottom left). The potential together with the solution still have the mirror symmetry with respect to the vertical axis. If we now increase the parameter beyond its critical value $\lambda = 0$, the origin becomes unstable as can be seen in fig. 11 (bottom second

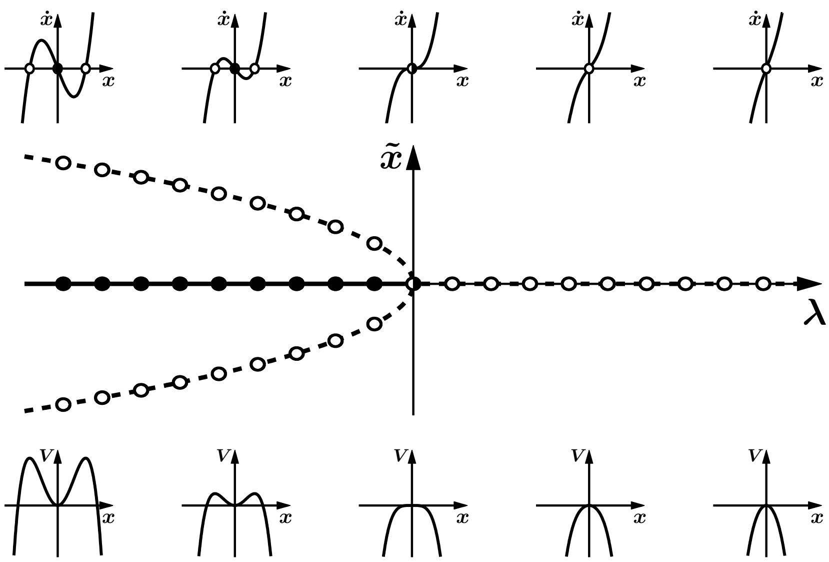

Fig. 12 Subcritical pitchfork bifurcation: a stable and two unstable fixed points collide and the former attractor becomes a repeller. Top: phase space plots; middle: bifurcation diagram; bottom: potential functions.

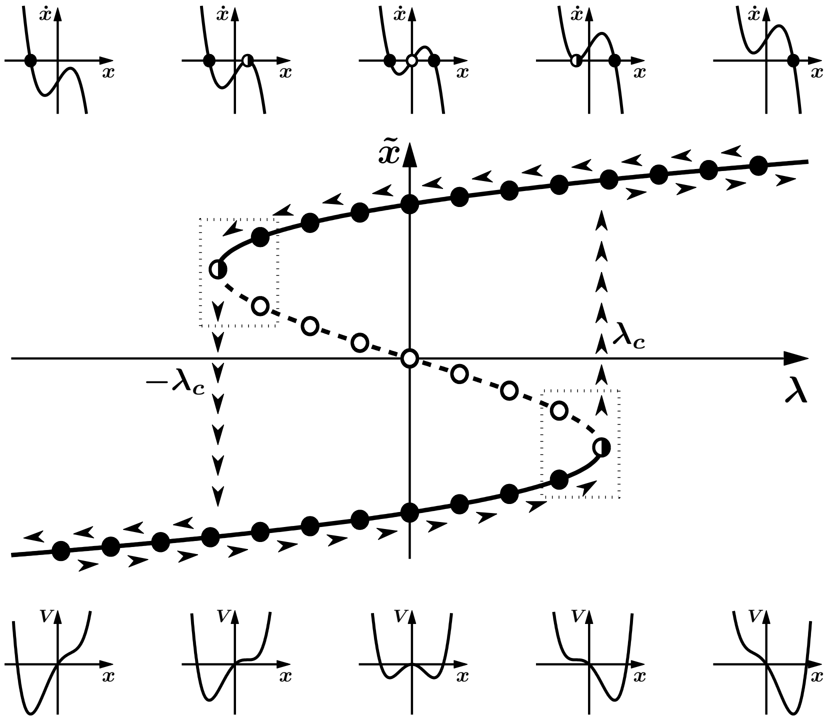

Fig. 13 A system showing hysteresis. Depending on whether the parameter is increased from large negative or decreased from large positive values the switch occurs at $\lambda = \lambda_c$ or $\lambda = -\lambda_c$, respectively. The bifurcation is not a basic type but consists of two saddle-node bifurcations indicated by the dotted rectangles. Top: phase space plots; middle: bifurcation diagram; bottom: potential functions.

from right). Now the slightest perturbation will move the ball to the left or right where the slope is finite and it will settle down in one of the new minima (fig. 11 (bottom right)). At this point, the potential plus solution is not symmetric anymore, the symmetry of the system has been broken by the solution. This phenomenon, called *spontaneous symmetry breaking,* is found in many systems in nature.

Subcritical Pitchfork Bifurcation

The equation governing the subcritical pitchfork bifurcation is given by

$$\dot{x} = \lambda x + x^3 \quad \rightarrow \quad \tilde{x}_1 = 0, \;\; \tilde{x}_{2,3} = \pm\sqrt{-\lambda} \tag{12}$$

and and its diagrams are shown in fig. 12. As in the supercritical case the origin is stable for negative values of λ and becomes unstable when the parameter exceeds $\lambda = 0$. Two additional fixed points exist for negative parameter values at $\tilde{x} = \pm\sqrt{-\lambda}$ and they are repellers.

System with Hysteresis

As we have seen before the system

$$\dot{x} = \lambda + x - x^3 \tag{13}$$

shows hysteresis, a phenomenon best visualized in the bifurcation diagram in fig. 13. If we start at a parameter value below the critical value $-\lambda_c$ and increase λ slowly, we will follow a path indicated by the arrows below the lower solid branch of stable fixed points in the bifurcation diagram. When we reach $\lambda = \lambda_c$ this branch does not continue and the system has to jump to the upper branch. Similarly, if we start at a large positive value of λ and decrease the parameter, we will stay on the upper branch of stable fixed points until we reach the point $-\lambda_c$ from where there is no smooth way out and a discontinuous switch to the lower branch occurs.

It is important to realize that (13) is not a basic bifurcation type. In fact, it consists of two saddle-node bifurcations indicated by the dotted rectangles in fig. 13.

2 Two-Dimensional Systems

Two-dimensional dynamical systems can be represented by either a single differential equation of second order, which contains a second derivative with respect to time, or by two equations of first order. In general, a second order system can always be expressed as two first order equations, but most first order systems cannot be written as a single second order equation

$$\ddot{x} + f(x,\dot{x}) = 0 \quad \rightarrow \quad \begin{cases} \dot{x} = y \\ \dot{y} = -f(x, y = \dot{x}) \end{cases} \tag{14}$$

2.1 Linear Systems and their Classification

A general linear two-dimensional system is given by

$$\begin{aligned} \dot{x} &= ax + by \\ \dot{y} &= cx + dy \end{aligned} \tag{15}$$

and has a fixed at the origin $\tilde{x} = 0$, $\tilde{y} = 0$.

The Pedestrian Approach

One may ask the question whether it is possible to decouple this system somehow, such that $\dot{x}$ only depends on x and $\dot{y}$ only on y. This would mean that we have two one-dimensional equations instead of a two-dimensional system. So we try

$$\begin{matrix} \dot{x} = \lambda x \\ \dot{y} = \lambda y \end{matrix} \quad \rightarrow \quad \begin{matrix} ax + by = \lambda x \\ cx + dy = \lambda y \end{matrix} \quad \rightarrow \quad \begin{matrix} (a - \lambda)x + by = 0 \\ cx + (d - \lambda)y = 0 \end{matrix} \tag{16}$$

where we have used (15) and obtained a system of equations for x and y. Now we are trying to solve this system

$$\begin{aligned} y = -\frac{a-\lambda}{b}x \quad &\rightarrow \quad cx - \frac{(a-\lambda)(d-\lambda)}{b}x = 0 \\ &\rightarrow \quad [\underbrace{(a-\lambda)(d-\lambda) - bc}_{=0}]x = 0 \end{aligned} \tag{17}$$

From the last term it follows that $x = 0$ is a solution, in which case form the first equation follows $y = 0$. However, there is obviously a second way how this system of equation can be solved, namely, if the under-braced term inside the brackets vanishes. Moreover, this term contains the parameter λ, which we have introduced in a kind of ad hoc fashion above, and now can be determined such that this term actually vanishes

$$\begin{aligned} (a-\lambda)(d-\lambda) - bc = 0 \quad &\rightarrow \quad \lambda^2 - (a+d)\lambda + ad - bc = 0 \\ &\rightarrow \quad \lambda_{1,2} = \tfrac{1}{2}\{a + d \pm \sqrt{(a+d)^2 - 4(ad-bc)}\} \end{aligned} \tag{18}$$

For simplicity, we assume $a = d$, which leads to

$$\lambda_{1,2} = a \pm \frac{1}{2}\sqrt{4a^2 - 4a^2 + 4bc} = a \pm \sqrt{bc} \tag{19}$$

As we know λ now, we can go back to the first equation in (17) and calculate y

$$y = -\frac{a-\lambda}{b}x = -\frac{a - (a \pm \sqrt{bc})}{b}x = \pm\sqrt{\frac{c}{b}}\,x \tag{20}$$

So far, so good but we need to figure out what this all means. In the first step we assumed $\dot{x} = \lambda x$ and $\dot{y} = \lambda y$. As we know from the one-dimensional case, such systems are stable for $\lambda < 0$ and unstable for $\lambda > 0$. During the calculations above we found two possible values for lambda, $\lambda_{1,2} = a \pm \sqrt{bc}$, which depend on the parameters of the dynamical system $a = d$, b and c. Either of them can be positive or negative, in fact if the product bc is negative, the λs can even be complex. For now we are going to exclude the latter case, we shall deal with it later. In addition, we have also found a relation between x and y for each of the values of λ, which is given by (20). If we plot y as a function of x (20) defines two straight lines through the origin with slopes of $\pm\sqrt{c/b}$, each of these lines corresponds to one of the values of lambda and the dynamics along these lines is given by $\dot{x} = \lambda x$ and $\dot{y} = \lambda y$. Along each of these lines the system can either approach the origin from both sides, in which cases it is called a stable direction or move away from it, which means the direction is unstable. Moreover, these are the only directions in the xy-plane where the dynamics evolves along straight lines and therefore built up a skeleton from which other trajectories can be easily constructed. Mathematically, the λs are called the eigenvalues and the directions represent the eigenvectors of the coefficient matrix as we shall see next.

There are two important descriptors of a matrix in this context, the trace and the determinant. The former is given by the sum of the diagonal elements $t_r = a + d$ and the latter, for a 2×2 matrix, is the difference between the products of the upper-left times lower-right and upper-right times lower-left elements $d_{et} = ad - bc$.

The Matrix Approach

Any two-dimensional linear system can be written in matrix form

$$\dot{\mathbf{x}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mathbf{x} = L\,\mathbf{x} \quad \rightarrow \quad \tilde{\mathbf{x}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{21}$$

with a fixed point at the origin. If a linear system's fixed point is not at the origin a coordinate transformation can be applied that shifts the fixed point such that (21) is fulfilled. The eigenvalues of L can be readily calculated and it is most convenient to express them in terms of the trace and determinant of L

$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = \lambda^2 - \lambda \underbrace{(a+d)}_{\text{trace } t_r} + \underbrace{ad-bc}_{\text{determinant } d_{et}} = 0 \tag{22}$$

$$\begin{aligned} \rightarrow \quad \lambda_{1,2} &= \tfrac{1}{2}\{a+d \pm \sqrt{(a+d)^2 - 4\,(ad-bc)}\} \\ &= \tfrac{1}{2}\{t_r \pm \sqrt{t_r^2 - 4\,d_{et}}\} \end{aligned} \tag{23}$$

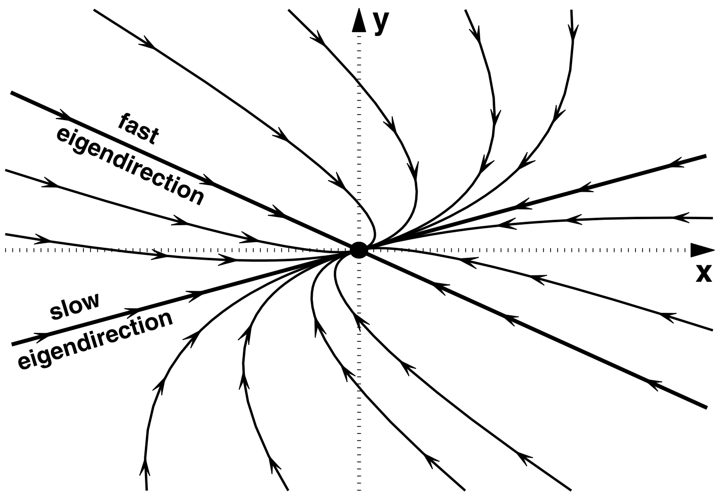

Fig. 14 Phase space portrait for the stable node.

Depending on whether the discriminant $t_r^2 - 4d_{et}$ in (23) is bigger or smaller than zero, the eigenvalues $\lambda_{1,2}$ will be real or complex numbers, respectively.

$\mathbf{t_r^2 - 4d_{et} > 0 \quad \rightarrow \quad \lambda_{1,2} \in \mathbb{R}}$

If both eigenvalues are negative, the origin is a stable fixed point, in this case called a *stable node*. An example of trajectories in the two-dimensional phase space is shown in fig. 14. We assume the two eigenvalues to be unequal, $\lambda_1 < \lambda_2$ and both smaller than zero. Then, the only straight trajectories are along the eigendirections which are given by the eigenvectors of the system. All other trajectories are curved as the rate of convergence is different for the two eigendirections depending on the corresponding eigenvalues. As we assumed $\lambda_1 < \lambda_2$ the trajectories approach the fixed point faster along the direction of the eigenvector $\mathbf{v}^{(1)}$ which corresponds to λ_1 and is therefore called the *fast eigendirection*. In the same way, the direction related to λ_2 is called the *slow eigendirection*.

Correspondingly, for the phase space plot when both eigenvalues are positive the flow, as indicated by the arrows in fig. 14, is reversed and leads away from the fixed point which is then called an *unstable node*.

For the degenerate case, with $\lambda_1 = \lambda_2$ we have a look at the system with

$$L = \begin{pmatrix} -1 & b \\ 0 & -1 \end{pmatrix} \quad \rightarrow \quad \lambda_{1,2} = -1 \tag{24}$$

The eigenvectors are given by

$$\begin{pmatrix} -1 & b \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \quad \rightarrow \quad \begin{aligned} -v_1 + b\,v_2 &= -v_1 \\ -v_2 &= -v_2 \end{aligned} \quad \rightarrow \quad b\,v_2 = 0 \tag{25}$$

For $b \neq 0$ the only eigendirection of L is the horizontal axis with $v_2 = 0$. The fixed point is called a *degenerate node* and its phase portrait shown in fig. 15 (left). If $b = 0$ any vector is an eigenvector and the trajectories are straight lines pointing towards or away from the fixed point depending on the sign of the eigenvalues. The phase space portrait for this situation is shown in fig. 15 (right) and the fixed point is for obvious reasons called a *star node* .

If one of the eigenvalues is positive and the other negative, the fixed point at the origin is half-stable and called a *saddle point*. The eigenvectors define the directions where the flow in phase space is pointing towards the fixed point, the so-called *stable direction*, corresponding to the negative eigenvalue, and away from the fixed point, the *unstable direction*, for the eigenvector with a positive eigenvalue. A typical phase space portrait for a saddle point is shown in fig. 16.

$\mathbf{t_r^2 - 4d_{et} < 0 \quad \rightarrow \quad \lambda_{1,2} \in \mathbb{C} \quad \rightarrow \quad \lambda_2 = \lambda_1^*}$

If the discriminant $t_r^2 - 4d_{et}$ in (23) is negative the linear two-dimensional system has a pair of complex conjugate eigenvalues. The stability of the fixed point is then

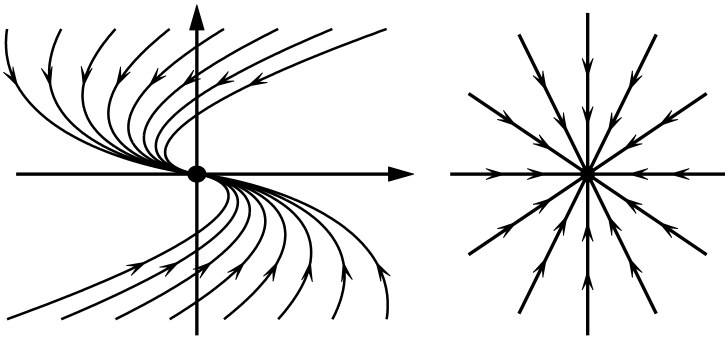

Fig. 15 Degenerate case where the eigenvalues are the same. The degenerate node (left) has only one eigendirection, the star node (right) has infinitely many.

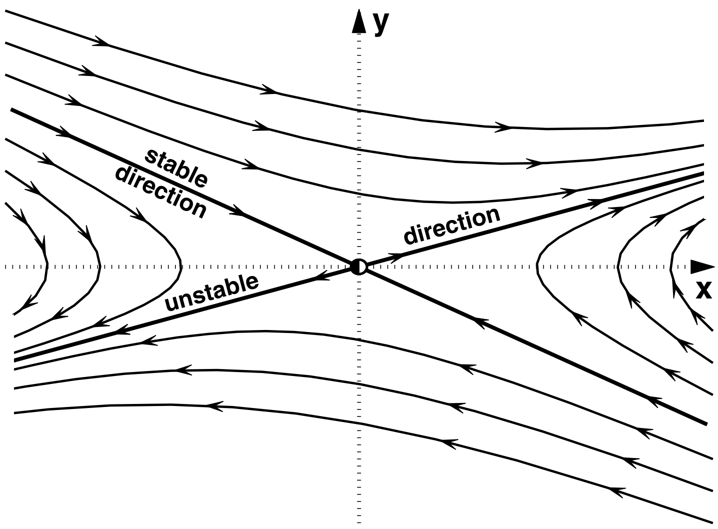

Fig. 16 If the eigenvalues have different signs $\lambda_1 \lambda_2 < 0$ the fixed point at the origin is half-stable and called a saddle point.

determined by the real part of the eigenvalues given as the trace of the coefficient matrix L in (21). The trajectories in phase space are spiraling towards or away from the origin as a *stable spiral* for a negative real part of the eigenvalue or an *unstable spiral* if the real part is positive as shown in fig. 17 left and middle, respectively. A special case exists when the real part of the eigenvalues vanishes $t_r = 0$. As can be seen in fig. 17 (right) the trajectories are closed orbits. The fixed point at the origin is neutrally stable and called a *center*.

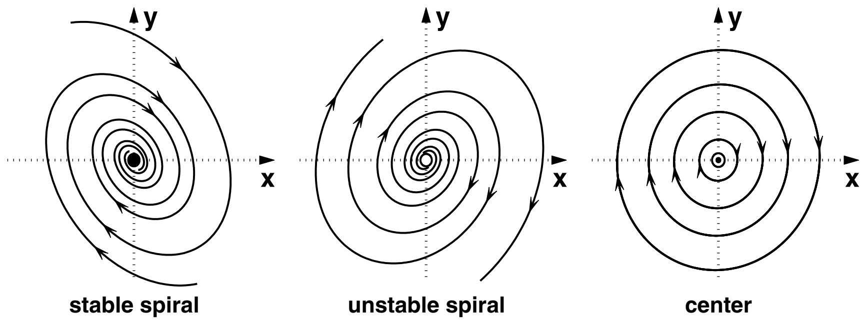

Fig. 17 For complex eigenvalues the trajectories in phase space are stable spirals if their real part is negative (left) and unstable spirals for a positive real part (middle). If the real part of the eigenvalues vanishes the trajectories are closed orbits around the origin, which is then a neutrally stable fixed point called a center (right).

To summarize these findings, we can now draw a diagram in a plane as shown in fig. 18, where the axes are the determinant d_{et} and trace t_r of the linear matrix L that provides us with a complete classification of the linear dynamical systems in two dimensions.

On the left of the vertical axis ($d_{et} < 0$) are the saddle points. On the right ($d_{et} > 0$) are the centers on the horizontal axis ($t_r = 0$) with unstable and stable spirals located above and below, respectively. The stars and degenerate nodes are along the parabola $t_r^2 = 4d_{et}$ that separates the spirals from the stable and unstable nodes.

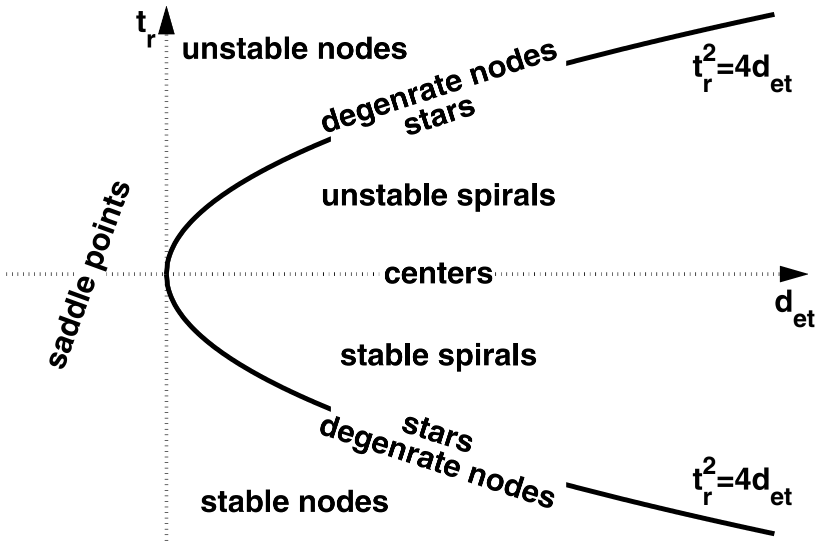

Fig. 18 Classification diagram for two-dimensional linear systems in terms of the trace t_r and determinant d_{et} of the linear matrix.

2.2 Nonlinear Systems

In general a two dimensional dynamical system is given by

$$\dot{x} = f(x,y) \qquad \dot{y} = g(x,y) \tag{26}$$

In the one-dimensional systems the fixed points were given by the intersection between the function in phase space and the horizontal axis. In two dimensions we can have graphs that define the locations $\dot{x} = f(x,y) = 0$ and $\dot{y} = g(x,y) = 0$, which are called the *nullclines* and are the location in the *xy*-plane where the tangent to the trajectories is vertical or horizontal, respectively. Fixed points are located at the intersections of the nullclines. We have also seen in one-dimensional systems that the stability of the fixed points is given by the slope of the function at the fixed point. For the two-dimensional case the stability is also related to derivatives but now there is more than one, there is the so-called *Jacobian matrix*, which has to be evaluated at the fixed points

$$J = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} \tag{27}$$

The eigenvalues and eigenvectors of this matrix determine the nature of the fixed points, whether it is a node, star, saddle, spiral or center and also the dynamics in its vicinity, which is best shown in a detailed example.

Detailed Example

We consider the two-dimensional system

$$\dot{x} = f(x,y) = y - y^3 = y(1-y^2), \quad \dot{y} = g(x,y) = -x - y^2 \tag{28}$$

for which the nullclines are given by

$$\begin{aligned} \dot{x} = 0 \quad &\rightarrow \quad y = 0 \quad \text{and} \quad y = \pm 1 \\ \dot{y} = 0 \quad &\rightarrow \quad y = \pm\sqrt{-x} \end{aligned} \tag{29}$$

The fixed points are located at the intersections of the nullclines

$$\tilde{\mathbf{x}}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \tilde{\mathbf{x}}_{2,3} = \begin{pmatrix} -1 \\ \pm 1 \end{pmatrix} \tag{30}$$

We determine the Jacobian of the system by calculating the partial derivatives

$$J = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} = \begin{pmatrix} 0 & 1-3y^2 \\ -1 & -2y \end{pmatrix} \tag{31}$$

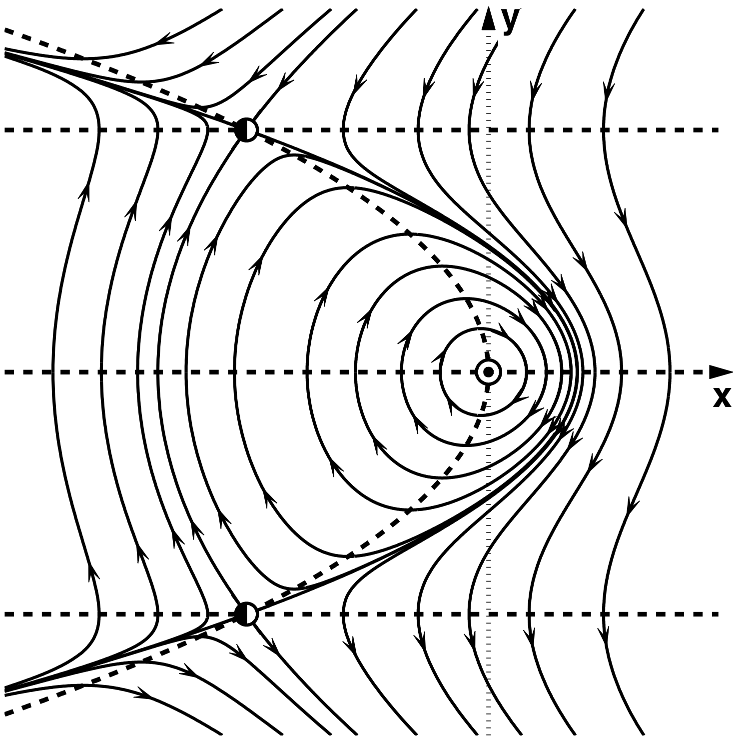

Fig. 19 Phase space diagram for the system (28).

A phase space plot for the system (28) is shown in fig. 19. The origin is a center surrounded by closed orbits with flow in the clockwise direction. This direction is readily determined by calculating the derivatives close to the origin

$$\dot{\mathbf{x}} = \begin{pmatrix} y - y^3 \\ -x - y^2 \end{pmatrix} \quad \text{at} \quad \mathbf{x} = \begin{pmatrix} 0.1 \\ 0 \end{pmatrix} \quad \rightarrow \quad \dot{\mathbf{x}} = \begin{pmatrix} 0 \\ -0.1 \end{pmatrix} \quad \rightarrow \quad \text{clockwise}$$

The slope of the trajectories at the two saddle points is given by the direction of their eigenvectors, and whether a particular direction is stable or unstable is determined by the corresponding eigenvalues. The two saddles are connected by two trajectories and such connecting trajectories between two fixed points are called *heteroclinic orbits*. The dashed horizontal lines through the fixed points and the dashed parabola which opens to the left are the nullclines where the the trajectories are either vertical or horizontal.

Second Example: Homoclinic Orbit

In the previous example we encountered heteroclinic orbits, which are trajectories that leave a fixed point along one of its unstable directions and approach another fixed point along a stable direction. In a similar way it is also possible that the trajectory returns along a stable direction to the fixed point it originated from. Such a closed trajectory that starts and ends at the same fixed point is correspondingly called a *homoclinic orbit*. To be specific we consider the system

$$\begin{aligned} \dot{x} &= y - y^2 = y(1-y) \\ \dot{y} &= x \end{aligned} \qquad \rightarrow \quad \tilde{\mathbf{x}}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \tilde{\mathbf{x}}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{32}$$

with the Jacobian matrix

$$J = \begin{pmatrix} 0 & 1-2y \\ 1 & 0 \end{pmatrix} \quad \rightarrow \quad J\,(\tilde{\mathbf{x}}_{1,2}) = \begin{pmatrix} 0 & \pm 1 \\ 1 & 0 \end{pmatrix} \tag{33}$$

From $t_r[J\,(\tilde{\mathbf{x}}_1)] = 0$ and $d_{et}[J\,(\tilde{\mathbf{x}}_1)] = -1$ we identify the origin as a saddle point. In the same way with $t_r[J\,(\tilde{\mathbf{x}}_2)] = 0$ and $d_{et}[J\,(\tilde{\mathbf{x}}_2)] = 1$ the second fixed point is classified as a center.

The eigenvalues and eigenvectors are readily calculated

$$\tilde{\mathbf{x}}_1: \;\; \lambda_{1,2}^{(1)} = \pm\sqrt{2}, \quad \mathbf{v}_{1,2}^{(1)} = \begin{pmatrix} 1 \\ \pm\sqrt{2} \end{pmatrix} \qquad \tilde{\mathbf{x}}_2: \;\; \lambda_{1,2}^{(2)} = \pm i\sqrt{2} \tag{34}$$

The nullclines are given by $y = 0$, $y = 1$ (vertical) and $x = 0$ (horizontal).

A phase space plot for the system (32) is shown in fig. 20 where the fixed point at the origin has a homoclinic orbit. The trajectory is leaving $\tilde{\mathbf{x}}_1$ along the unstable direction, turning around the center $\tilde{\mathbf{x}}_2$ and returning along the stable direction of the saddle.

2.3 Limit Cycles

A limit cycle, the two-dimensional analogon of a fixed point, is an *isolated closed* trajectory. Consequently, limit cycles exist with the flavors *stable*, *unstable* and *half-stable* as shown in fig. 21. A stable limit cycle attracts trajectories from both its outside and its inside, whereas an unstable limit cycle repels trajectories on both sides. There also exist closed trajectories, called half-stable limit cycles, which attract the trajectories from one side and repel those on the other. Limit cycles are inherently nonlinear objects and must not be mixed up with the centers found in the previous section in linear systems when the real parts of both eigenvalues vanish. These centers are not isolated closed trajectories, in fact there is always another closed trajectory infinitely close nearby. Also all centers are neutrally stable, they are neither attracting nor repelling.

From fig. 21 it is intuitively clear that inside a stable limit cycle, there must be an unstable fixed point or an unstable limit cycle, and inside an unstable limit cycle there is a stable fixed point or a stable limit cycle. In fact, this intuition will guide us to a new and one of the most important bifurcation types: the *Hopf bifurcation*.

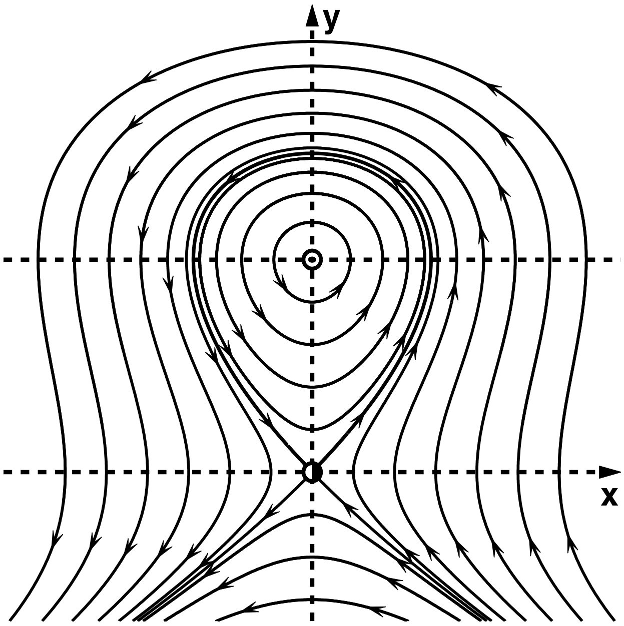

Fig. 20 Phase space diagram with a homoclinic orbit.

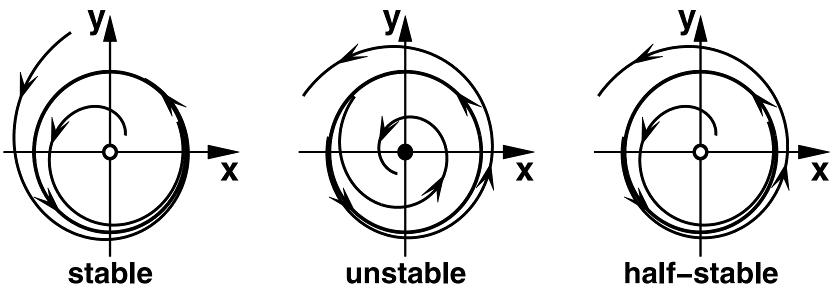

Fig. 21 Limit cycles attracting or/and repelling neighboring trajectories.

2.4 Hopf Bifurcation

We consider the dynamical system

$$\dot{\xi} = \mu\,\xi - \xi\,|\,\xi\,|^2 \quad \text{with} \quad \mu,\xi \in \mathbb{C} \tag{35}$$

where both the parameter μ and the variable ξ are complex numbers. There are essentially two ways in which complex numbers can be represented

1. Cartesian representation: $\xi = x + iy$ for which (35) takes the form

$$\begin{aligned} \dot{x} &= \varepsilon x - \omega y - x(x^2+y^2) \\ \dot{y} &= \varepsilon x + \omega y - y(x^2+y^2) \end{aligned} \tag{36}$$

   after assuming $\mu = \varepsilon + i\omega$ and splitting into real and imaginary part;

2. Polar representation: $\xi = r\,e^{i\varphi}$ and (35) becomes

$$\dot{r} = \varepsilon r - r^3 \qquad \dot{\varphi} = \omega \tag{37}$$

Rewriting (35) in a polar representation leads to a separation of the complex equation not into a coupled system as in the cartesian case (36) but into two uncoupled first order differential equations, which both are quite familiar. The second equation for the phase φ can readily be solved, $\varphi(t) = \omega t$, the phase is linearly increasing with time, and, as φ is a cyclic quantity, has to be taken modulo 2π. The first equation is the well-known cubic equation (7) this time simply written in the variable r instead of x. As we have seen earlier, this equation has a single stable fixed point $r = 0$ for $\varepsilon < 0$ and undergoes a pitchfork bifurcation at $\varepsilon = 0$, which turns the fixed point $r = 0$ unstable and gives rise to two new stable fixed points at $r = \pm\sqrt{\varepsilon}$. Interpreting r as the radius of the limit cycle, which has to be greater than zero, we find that a stable limit cycle arises from a fixed point, when ε exceeds its critical value $\varepsilon = 0$.

To characterize the behavior that a stable fixed point switches stability with a limit cycle in a more general way, we have a look at the linear part of (35) in its cartesian form

$$\dot{\xi} = \mu\xi = (\varepsilon + i\omega)(x + iy) \quad \rightarrow \quad \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \varepsilon & -\omega \\ \omega & \varepsilon \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{38}$$

The eigenvalues λ for the matrix in (38) are found from the characteristic polynomial

$$\begin{aligned} &\begin{vmatrix} \varepsilon - \lambda & -\omega \\ \omega & \varepsilon - \lambda \end{vmatrix} = \lambda^2 - 2\varepsilon\lambda + \varepsilon^2 + \omega^2 \\ &\qquad\qquad \rightarrow \quad \lambda_{1,2} = \varepsilon \pm \tfrac{1}{2}\sqrt{4\varepsilon^2 - 4\varepsilon^2 - 4\omega^2} = \varepsilon \pm i\omega \end{aligned} \tag{39}$$

A plot of $\Im(\lambda)$ versus $\Re(\lambda)$ is shown in fig. 22 for the system we discussed here on the left, and for a more general case on the right. Such a qualitative change in a dynamical system where a pair of complex conjugate eigenvalues crosses the vertical axis we call a *Hopf bifurcation*, which is the most important bifurcation type for a system that switches from a stationary state at a fixed point to oscillation behavior on a limit cycle.

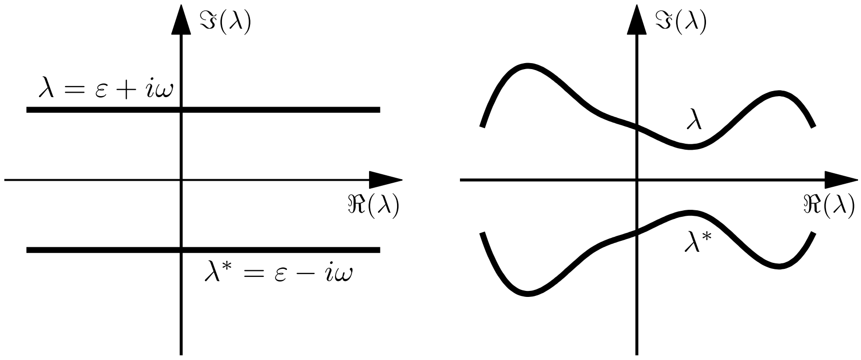

Fig. 22 A Hopf bifurcation occurs in a system when a pair of complex conjugate eigenvalues crosses the imaginary axis. For (39) the imaginary part of ε is a constant ω (left). A more general example is shown on the right.

2.5 Potential Functions in Two-Dimensional Systems

A two-dimensional system of first order differential equations of the form

$$\dot{x} = f(x,y) \qquad \dot{y} = g(x,y) \tag{40}$$

has a potential and is called a *gradient system* if there exists a scalar function of two variables $V(x,y)$ such that

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix} = -\begin{pmatrix} \dfrac{\partial V(x,y)}{\partial x} \\ \dfrac{\partial V(x,y)}{\partial y} \end{pmatrix} \tag{41}$$

is fulfilled. As in the one-dimensional case the potential function $V(x,y)$ is monotonically decreasing as time evolves, in fact, the dynamics follows the negative gradient and therefore the direction of steepest decent along the two-dimensional surface This implies that a gradient system cannot have any closed orbits or limit cycles.

An almost trivial example for a two-dimensional system that has a potential is given by

$$\dot{x} = -\frac{\partial V}{\partial x} = -x \qquad \dot{y} = -\frac{\partial V}{\partial x} = y \tag{42}$$

Technically, (42) is not even two-dimensional but two one-dimensional systems that are uncoupled. The eigenvalues and eigenvectors can easily be guessed as $\lambda_1 = -1$, $\lambda_2 = 1$ and $\mathbf{v}^{(1)} = (1,0)$, $\mathbf{v}^{(2)} = (0,1)$ defining the x-axis as a stable and the y-axis as an unstable direction. Applying the classification scheme, with $t_r = 0$ and $d_{et} = -1$ the origin is identified as a saddle. It is also easy to guess the potential function $V(x,y)$ for (42) and verify the guess by taking the derivatives with respect to x and y

$$V(x,y) = \frac{1}{2}x^2 - \frac{1}{2}y^2 \quad \rightarrow \quad \frac{\partial V}{\partial x} = x = -\dot{x} \qquad \frac{\partial V}{\partial y} = -y = -\dot{y} \tag{43}$$

A plot of this function in shown in fig. 23 (left). White lines indicate equipotential locations and a set of trajectories is plotted in black. The trajectories are following the negative gradient of the potential and therefore intersect the equipotential lines at a right angle. From the shape of the potential function on the left it is most evident why fixed points in two dimensions with a stable and an unstable direction are called saddles.

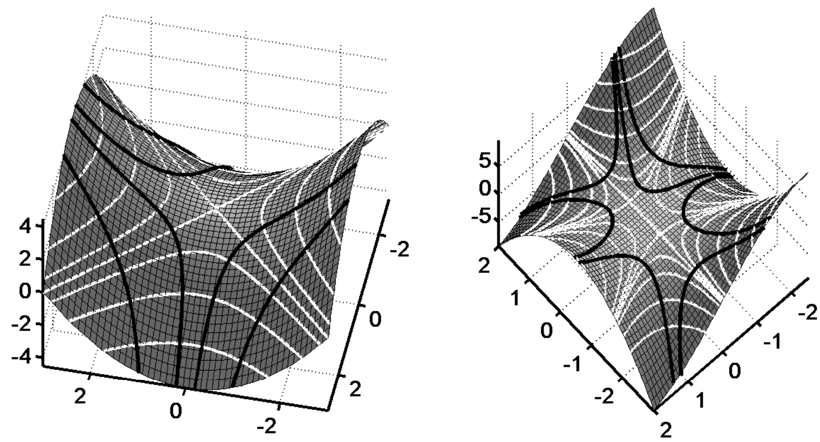

Fig. 23 Potential functions for a saddle (42) (left) and for the example given by (46) (right). Equipotential lines are plotted in white and a set of trajectories in black. As the trajectories follow the negative gradient of the potential they intersect the lines of equipotential at a right angle.

It is easy to figure out whether a specific two-dimensional system is a gradient system and can be derived from a scalar potential function. A theorem states that a potential exists if and only if the relation

$$\frac{\partial f(x,y)}{\partial y} = \frac{\partial g(x,y)}{\partial x} \tag{44}$$

is fulfilled. We can easily verify that (42) fulfills this condition

$$\frac{\partial f(x,y)}{\partial y} = -\frac{\partial x}{\partial y} = \frac{\partial g(x,y)}{\partial x} = \frac{\partial y}{\partial x} = 0 \tag{45}$$

However, in contrast to one-dimensional systems, which all have a potential, two-dimensional gradient systems are more the exception than the rule.

As a second and less trivial example we discuss the system

$$\dot{x} = y + 2xy \qquad \dot{y} = x + x^2 - y^2 \tag{46}$$

First we check whether (44) is fulfilled and (46) can indeed be derived from a potential

$$\frac{\partial f(x,y)}{\partial y} = \frac{\partial(y+2xy)}{\partial y} = \frac{\partial g(x,y)}{\partial x} = \frac{\partial(x+x^2-y^2)}{\partial x} = 1+2x \tag{47}$$

In order to find the explicit form of the potential function we first integrate $f(x,y)$ with respect to x, and $g(x,y)$ with respect to y

$$\begin{aligned} \dot{x} = f(x,y) = -\frac{\partial V}{\partial x} \quad &\rightarrow \quad V(x,y) = -xy - x^2y + c_x(y) \\ \dot{y} = g(x,y) = -\frac{\partial V}{\partial y} \quad &\rightarrow \quad V(x,y) = -xy - x^2y + \tfrac{1}{3}y^3 + c_y(x) \end{aligned} \tag{48}$$

As indicated the integration "constant" c_x for the x integration is still dependent on the the variable y and vice versa for c_y. These constants have to be chosen such that the potential $V(x,y)$ is the same for both cases, which is evidently fulfilled by choosing $c_x(y) = \frac{1}{3}y^3$ and $c_y(x) = 0$. A plot of $V(x,y)$ is shown in fig. 23 (right). Equipotential lines are shown in white and some trajectories in black. Again the trajectories follow the gradient of the potential and intersect the contour lines at a right angle.

2.6 Oscillators

Harmonic Oscillator

The by far best known two-dimensional dynamical system is the harmonic oscillator given by

$$\ddot{x} + 2\gamma\dot{x} + \omega^2 x = 0 \quad \text{or} \quad \begin{cases} \dot{x} = y \\ \dot{y} = -2\gamma y - \omega^2 x \end{cases} \tag{49}$$

Here ω is the angular velocity, sometimes referred to in a sloppy way as frequency, γ is the damping constant and the factor 2 allows for avoiding fractions in some formulas later on. If the damping constant vanishes, the trace of the linear matrix is zero and its determinant ω^2, which classifies the fixed point at the origin as a center. The generals solution of (49) in this case is given by a superposition of a cosine and sine function

$$x(t) = a\cos\omega t + b\sin\omega t \tag{50}$$

where the open parameters a and b have to be determined from initial conditions, displacement and velocity at $t = 0$ for instance.

If the damping constant is finite, the trance longer vanishes and the phase space portrait is a stable or unstable spiral depending on the sign of γ. For $\gamma > 0$ the time series is a damped oscillation (unless the damping gets really big, a case we leave as an exercise for the reader) and for $\gamma < 0$ the amplitude increases exponentially,

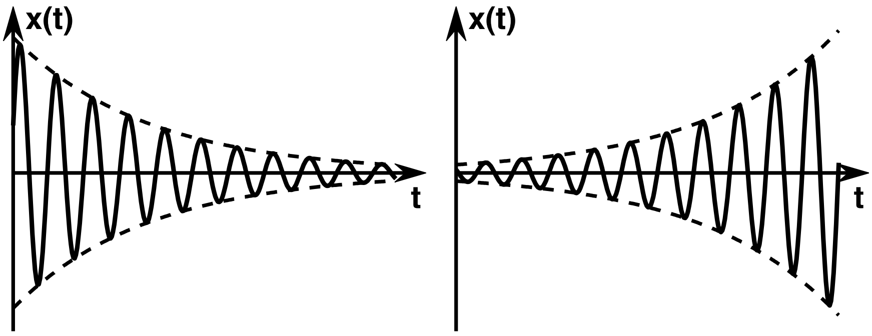

Fig. 24 Examples for "damped" harmonic oscillations for the case of positive damping $\gamma > 0$ (left) and negative damping $\gamma < 0$ (right).

both cases are shown in fig. 24. As it turns out, the damping not only has an effect on the amplitude but also on the frequency and the general solution of (49) reads

$$x(t) = e^{-\gamma t}\{a\cos\Omega t + b\sin\Omega t\} \quad \text{with} \quad \Omega = \sqrt{\gamma^2 - \omega^2} \tag{51}$$

Nonlinear Oscillators

As we have seen above harmonic (linear) oscillators do not have limit cycles, i.e. *isolated* closed orbits in phase space. For the linear center, there is always another orbit infinitely close by, so if a dynamics is perturbed it simply stays on the new trajectory and does not return to the orbit it originated from. This situation changes drastically as soon as we introduce nonlinear terms into the oscillator equation

$$\ddot{x} + \gamma\dot{x} + \omega^2 x + N(x,\dot{x}) = 0 \tag{52}$$

For the nonlinearites $N(x,\dot{x})$ there are infinitely many possibilities, even if we restrict ourselves to polynomials in x and $\dot{x}$. However, depending on the application there are certain terms that are more important than others, and certain properties of the system we are trying to model may give us hints, which nonlinearities to use or to exclude.

As an example we are looking for a nonlinear oscillator to describe the movements of a human limb like a finger, hand, arm or leg. Such movements are indeed limit cycles in phase space and if their amplitude is perturbed they return to the formerly stable orbit. For simplicity we assume that the nonlinearity is a polynomial in x and $\dot{x}$ up to third order, which means we can pick from the terms

$$\begin{array}{rl} \text{quadratic:} & x^2, x\dot{x}, \dot{x}^2 \\ \text{cubic:} & x^3, x^2\dot{x}, x\dot{x}^2, \dot{x}^3 \end{array} \tag{53}$$

For human limb movements, the flexion phase is in good approximation a mirror image of the extension phase. In the phase space portrait this is reflected by a point symmetry with respect to the origin or an invariance of the system under the transformation $x \rightarrow -x$ and $\dot{x} \rightarrow -\dot{x}$. In order to see the consequences of such an invariance we probe the system

$$\ddot{x} + \gamma\dot{x} + \omega^2 x + a x^2 + b x \dot{x} + c x^3 + d x^2 \dot{x} = 0 \qquad (54)$$

In (54) we substitute x by $-x$ and $\dot{x}$ by $-\dot{x}$ and obtain

$$-\ddot{x} - \gamma\dot{x} - \omega^2 x + a x^2 + b x \dot{x} - c x^3 - d x^2 \dot{x} = 0 \qquad (55)$$

Now we multiply (55) by -1

$$\ddot{x} + \gamma\dot{x} + \omega^2 x - a x^2 - b x \dot{x} + c x^3 + d x^2 \dot{x} = 0 \qquad (56)$$

Comparing (56) with (54) shows that the two equations are identical if and only if the coefficients a and b are zero. In fact, evidently any quadratic term cannot appear in an equation for a system intended to serve as a model for human limb movements as it breaks the required symmetry. From the cubic terms the two most important ones are those that have a main influence on the amplitude as we shall discuss in more details below. Namely, these nonlinearities are the so-called *van-der-Pol* term $x^2\dot{x}$ and the *Rayleigh* term $\dot{x}^3$.

Van-der-Pol Oscillator: $\mathbf{N(x,\dot{x}) = x^2\,\dot{x}}$

The van-der-Pol oscillator is given by

$$\ddot{x} + \gamma\dot{x} + \omega^2 x + \varepsilon x^2 \dot{x} = 0 \qquad (57)$$

which we can rewrite in the form

$$\ddot{x} + \underbrace{(\gamma + \varepsilon x^2)}_{\tilde{\gamma}}\dot{x} + \omega^2 x = 0 \qquad (58)$$

Equation (58) shows that for the van-der-Pol oscillator the damping "constant" $\tilde{\gamma}$ becomes time dependent via the amplitude x^2. Moreover, writing the van-der-Pol oscillator in the form (58) allows for an easy determination of the parameter values for γ and ε that can lead to sustained oscillations. We distinguish four cases:

$\gamma > 0,\ \varepsilon > 0$: The effective damping $\tilde{\gamma}$ is always positive. The trajectories are evolving towards the origin which is a stable fixed point;

$\gamma < 0,\ \varepsilon < 0$: The effective damping $\tilde{\gamma}$ is always negative. The system is unstable and the trajectories are evolving towards infinity;

$\gamma > 0,\ \varepsilon < 0$: For small values of the amplitude x^2 the effective damping $\tilde{\gamma}$ is positive leading to even smaller amplitudes. For large values of x^2 the effective damping $\tilde{\gamma}$ is negative leading a further increase in amplitude. The system evolves either towards the fixed point or towards infinity depending on the initial conditions;

$\gamma < 0,\ \varepsilon > 0:$ For small values of the amplitude x^2 the effective damping $\tilde{\gamma}$ is negative leading to an increase in amplitude. For large values of x^2 the effective damping $\tilde{\gamma}$ is positive and decreases the amplitude. The system evolves towards a stable limit cycle. Here we see a familiar scenario: without the nonlinearity the system is unstable ($\gamma < 0$) and moves away from the fixed point at the origin. As the amplitude increases the nonlinear damping ($\varepsilon > 0$) becomes an important player and leads to saturation at a finite value.

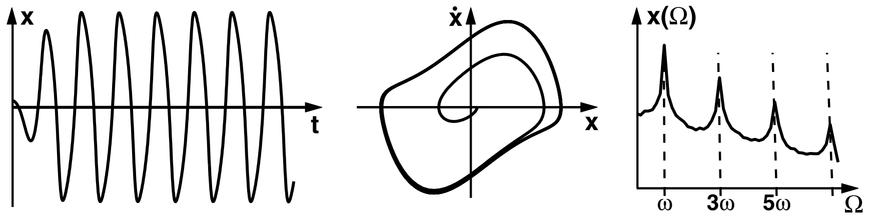

Fig. 25 The van-der-Pol oscillator: time series (left), phase space trajectory (middle) and power spectrum (right).

The main features for the van-der-Pol oscillator are shown in fig. 25 with the time series (left), the phase space portrait (middle) and the power spectrum (right). The time series is not a sine function but has a fast rising increasing flank and a more shallow slope on the decreasing side. Such time series are called *relaxation oscillations*. The trajectory in phase space is closer to a rectangle than a circle and the power spectrum shows pronounced peaks at the fundamental frequency ω and its odd higher harmonics $(3\omega, 5\omega \ldots)$.

Rayleigh Oscillator: $\mathbf{N(x,\dot{x}) = \dot{x}^3}$

The Rayleigh oscillator is given by

$$\ddot{x} + \gamma\dot{x} + \omega^2 x + \delta\dot{x}^3 = 0 \tag{59}$$

which we can rewrite as before

$$\ddot{x} + \underbrace{(\gamma + \delta\dot{x}^2)}_{\tilde{\gamma}}\dot{x} + \omega^2 x = 0 \tag{60}$$

In contrast to the van-der-Pol case the damping "constant" for the Rayleigh oscillator depends on the square of the velocity $\dot{x}^2$. Arguments similar to those used above lead to the conclusion that the Rayleigh oscillator shows sustained oscillations in the parameter range $\gamma < 0$ and $\delta > 0$.

As shown in fig. 26 the time series and trajectories of the Rayleigh oscillator also show relaxation behavior, but in this case with a slow rise and fast drop. As for

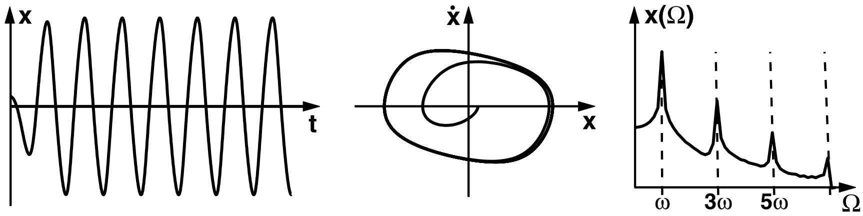

Fig. 26 The Rayleigh oscillator: time series (left), phase space trajectory (middle) and power spectrum (right).

the van-der-Pol oscillator, the phase space portrait is almost rectangular but the long and short axes are switched. Again the power spectrum has peaks at the fundamental frequency and the odd higher harmonics.

Taken by themselves neither the van-der-Pol nor Rayleigh oscillators are good models for human limb movement for at least two reasons even though they fulfill one requirement for a model: they have stable limit cycles. However, first, human limb movements are almost sinusoidal and their trajectories have a circular or elliptic shape. Second, it has been found in experiments with human subjects performing rhythmic limb movements that when the movement rate is increased, the amplitude of the movement decreases linearly with frequency. It can be shown that for the van-der-Pol oscillator the amplitude is independent of frequency and for the Rayleigh it decreases proportional to ω^{-2}, both in disagreement with the experimental findings.

Hybrid Oscillator: $\mathbf{N(x,\dot{x}) = \{x^2\dot{x}, \dot{x}^3\}}$

The hybrid oscillator has two nonlinearities, a van-der-Pol and a Rayleigh term and is given by

$$\ddot{x} + \gamma\dot{x} + \omega^2 x + \varepsilon x^2\dot{x} + \delta\dot{x}^3 = 0 \qquad (61)$$

which we can rewrite again

$$\ddot{x} + \underbrace{(\gamma + \varepsilon x^2 + \delta\dot{x}^2)}_{\tilde{\gamma}}\dot{x} + \omega^2 x = 0 \qquad (62)$$

The parameter range of interest is $\gamma < 0$ and $\varepsilon \approx \delta > 0$. As seen above, the relaxation phase occurs on opposite flanks for the van-der-Pol and Rayleigh oscillator. In combining both we find a system that not only has a stable limit cycle but also the other properties required for a model of human limb movement.

As shown in fig. 27 the time series for the hybrid oscillator is almost sinusoidal and the trajectory is elliptical. The power spectrum has a single peak at the fundamental frequency. Moreover, the relation between the amplitude and frequency is a linear decrease in amplitude when the rate is increased as shown schematically in fig. 28. Taken together, the hybrid oscillator is a good approximation for the trajectories of human limb movements.

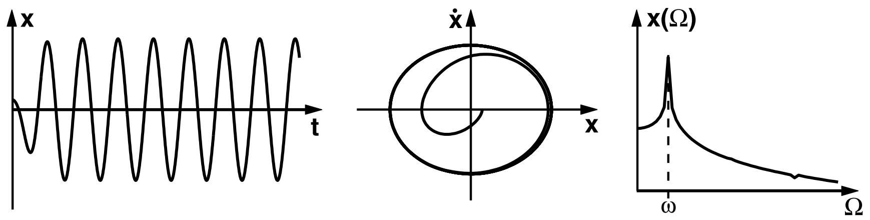

Fig. 27 The hybrid oscillator: time series (left), phase space trajectory (middle) and power spectrum (right).

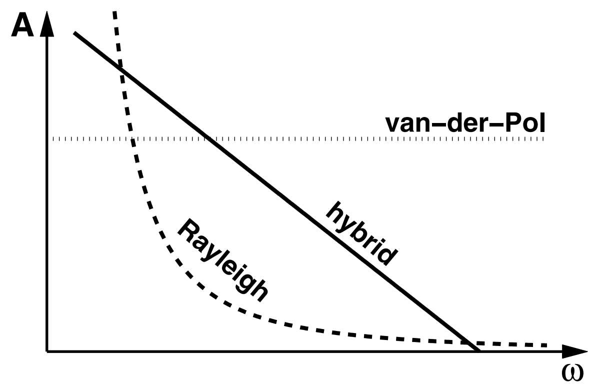

Fig. 28 Amplitude-frequency relation for the van-der-Pol (dotted), Rayleigh ($\sim \omega^{-2}$, dashed) and hybrid ($\sim -\omega$, solid) oscillator.

Beside the dynamical properties of the different oscillators, the important issue here, which we want to emphasize on, is the modeling strategy we have applied. Starting from a variety of quadratic and cubic nonlinearities in x and $\dot{x}$ we first used the symmetry between the flexion and extension phase of the movement to rule out any quadratic terms. Then we studied the influence of the van-der-Pol and Rayleigh terms on the time series, phase portraits and spectra. In combining these nonlinearities to the hybrid oscillator we found a dynamical system that is in agreement with the experimental findings, namely

- the trajectory in phase space is a stable limit cycle. If this trajectory is perturbed the system returns to its original orbit;
- the time series of the movement is sinusoidal and the phase portrait is elliptical;
- the amplitude of the oscillation decreases linearly with the movement frequency.

For the sake of completeness we briefly mention the influence of the two remaining cubic nonlinearities on the dynamics of the oscillator. The van-der-Pol and Rayleigh term have a structure of velocity times the square of location or velocity, respectively, which we have written as a new time dependent damping term. Similarly, the remaining terms $x\dot{x}^2$ and x^3 (the latter called a Duffing term) are of the form location times the square of velocity or location. These nonlinearities can be written as a time dependent frequency, leading to an oscillator equation with all cubic nonlinear terms

$$\ddot{x} + \underbrace{(\gamma + \varepsilon x^2 + \delta \dot{x}^2)}_{\tilde{\gamma}\ \text{damping}} \dot{x} + \underbrace{(\omega^2 + \alpha \dot{x}^2 + \beta x^2)}_{\tilde{\omega}^2\ \text{frequency}} x = 0 \quad (63)$$

Further Readings

Strogatz, S.H.: Nonlinear Dynamics and Chaos. Perseus Books Publishing, Cambridge (2000)

Haken, H.: Introduction and Advanced Topics. Springer, Berlin (2004)